Some Unusual Observations about an Achromatic Index of Graphs

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Abstract: This paper addresses some interesting observations about an achromatic index of graphs related to degree sequence of graphs, graphs & their complements & existing achromatic index of graphs.

Keywords: graphs, achromatic, colouring

Introduction:
A k-edge colouring of a simple graph G is assigning k colours to the edges of G so that no two adjacent edges receive same colours. If for each pair ti & tj of colours, there exist adjacent edges with these colours then the colouring is said to be complete. Let G be a simple graph. The achromatic index ψ(G) of a simple graph G is the maximum number of colours used in the edge colouring of G such that the colouring is complete. All though ψ(G) is known for some graphs but in general it is not known for arbitrary simple graphs. For Complete graph G of order n, ψ(G) is denoted either by A(Kn) or A(n).

Key words: Achromatic index, colouring of graphs, complete edge colouring, degree sequence

Let us begin with the following observations.

Observation 1:
It is known that for a simple graph G(p, q) whose degree sequence is d₁, d₂, d₃,…..dₚ

ψ(G) (ψ(G)-1) ≤ \( \sum d_i \) P₂

where the sum is taken over i=1 to p [²]

So it is natural to expect higher the number \( \sum d_i \) P₂ gives higher Achromatic Index of G

But this expectation is not true in general.

Consider the following graphs G₁

![Diagram of G₁](image1)

(numbers written on the edges are colours throughout this paper). It can be easily observe that the above colouring is the complete colouring of G₁

\[ \psi(G₁) \geq 5 \]

The degree sequence of G₁ is 2, 2, 2, 3, 4 hence

\[ \sum d_i \] P₂ = 28

\[ \psi(G₁)(\psi(G₁)-1) \leq 28 \]

\[ \psi(G₁) \leq 5 \]

from *1 & *2 we conclude \( \psi(G₁) = 5 \)

Now consider the following Graph G₂

![Diagram of G₂](image2)

The degree sequence of G₂ is 1, 3, 3, 3, 4 hence

\[ \sum d_i \] P₂ = 30

\[ \psi(G₂)(\psi(G₂)-1) \leq 30 \]

\[ \psi(G₂) \leq 6 \]

Now we will try to obtain a complete colouring of G₂ with five colours.

WLG BA=4 (we mean BA is coloured with the colour 4)
BE=3, BD=2, BC=1

Now we will think of inserting colour 5 in G₂

If CE=5 then the colour 2 cannot be made adjacent to the colour 5 as the choices for CD, DE will not establish the proper colouring of the graph G₂.

If CD=5 then the colour 3 cannot be made adjacent to the colour 5 as the choices for CE, DE will not establish the proper colouring of the graph G₂.

If DE=5 then the colour 1 cannot be made adjacent to the colour 5 as the choices for CD, CE will not establish the proper colouring of the graph G₂.

\[ \psi(G₂) \leq 4 \]

The following colouring shows \( \psi(G₂) = 4 \)

Though both graphs have same number of edges & same number of vertices & \[ \sum d_i \] P₂ of G₁ ≤ \[ \sum d_i \] P₂ of G₂, \[ \psi(G₁) \geq \psi(G₂) \]

Observation 2

Let H be a spanning sub graph of Kn. Then there may be any kind of inequality or equality between \( \psi(H) + \psi(H') \) & \( \psi(K_n) \)
as shown in the following cases.

case i) Let n=11 H be K₁₁ \cup \{ single isolated vertex \} therefore H’ is a spanning K₁,₁₀ tree of K₁₁
It is known that \[ \psi(K₁₁)=27, \psi(H)=10, \psi(H')=22 \]

\[ \psi(K₁₁) < \psi(H) + \psi(H') \]
It is naturally posing like $A(n+1) \leq A(n) + n$ which is a trivial result.\[1\]

Case ii) Let $n=5$ & $H, H^c$ are as shown below

Case iii) Let $n=5$ & $H$ be $K_4 \cup \{\text{single isolated vertex}\}$ therefore $H^c$ is a spanning $K_{1,4}$ tree of $K_n$

It is known that $\psi'(K_3) = 7$
It is easy to check $\psi'(H) = 3$, $\psi'(H^c) = 3$

$\therefore \psi'(K_n) \geq \psi'(H) + \psi'(H^c)$

Observation 3:
The bounds of achromatic index of complete graphs $K_n$ up to $n=30$ are given below\[1\]

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It can be observe that the condition $A(2n+1) \geq A(2n)[1 + 2n/(A(2n)+n(2n-1))]$ remains valid up to $n=13$, so if the condition is considered to be true then it tights some of the upper & lower bounds of achromatic indices of graphs.

For example substitute $n=8$ in the above condition
It gives $A(17) \geq A(16)[1+16/(A(16)+120)]$
$\therefore A(16) \leq A(17)/[1+16/(A(16)+120)]$
$\therefore A(16) \leq A(17)/1.09937888$
$\therefore A(16) \leq 51$
$\therefore U.B. of $K_{16}$ reduces to 51$

If we substitute $n=13$
$A(27) \geq A(26)[1+26/(A(26)+325)]$
$\therefore A(27) \geq 105x1.06$
$\therefore A(27) \geq 111$
$\therefore L.B. of $K_{27}$ rises to 111$

References:

i. R.E. Jamison, on the edge of achromatic numbers of graphs, Discrete Mathematics (1989) ,99-115
iii. V. N. Bhatnayak & M.Shanti Achromatic number of graphs & its complement Bulletin of the Bombay Mathematical Colloquim 6(1989) 9-14
iv. C. Vasudev, Graph Theory with Applications
v. Pruna Chandra Biswal, Discrete Mathematics & Graph Theory
vi. Richard Brualdi, Introductory Combinatorics
vii. J. A. Bondy, U.S.R.Murty, Graph Theory with applications
viii. Frank Harary, Graph Theory
ix. Herbert L. Cooper, The Spirit of “C”

x. V. J. Vernold, M.Venkatachalam,Ali M. M. Akbar, A Note On Achromatic Coloring Of Star Graph Families, Filomat23:3 (2009), 251-255