International Journal of Scientific Engineering and Technology (ISSN : 2277-1581)
www.ijset.com, Volume No.1, Issue No.3, pg:29-33 01 July 2012

Various Synchronization Schemes for Chaotic Dynamical Systems (A Classical Survey)

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***Abstract*—The phenomenon of chaotic system synchronization is very interesting because of its high potential of applications. In this paper results from various synchronization schemes in the form of a survey articles with a tutorial emphasis has been presented. Chaotic dynamical systems having features such as sensitive dependence on initial conditions and topological mixing along with periodic dense orbits. In order to optimize the results of synchronization is also a motivating factor for the study of this phenomenon.**

Keywords- local and global asymptotic stability, chaotic coupling, chaos synchronization, chaos control

I. INTRODUCTION

In 17th century the famous Dutch physicist Christian Huygens reported the first observation of synchronization of two pendulum clocks in 1665 [1]. The experimental as well as theoretical study of this phenomenon was started by Edward Appleton (1922) and Balthasar van der Pol (1927).They showed that by a weak external signal with slightly different frequency, the frequency of a triode generator can be entrained, or synchronized. Andronov and Vitt (1930), representatives of the Russian school, provided the development of synchronization theory. Mandelshtam and Papaleksi (1947), studied about the $n:m$ external synchronization. Mutual synchronization of two weakly nonlinear oscillators was analytically treated by Mayer (1935) and Gaponov (1936) after that relaxation oscillators were studied by Bremsen and Feinberg (1941). Finally in the monographs of Teodorchik (1952), Hayashi (1964), Malakhov (1968), Blekhman (1971, 1981), Landa (1980, 1996), Romanovsky et al. (1984) and Kuramoto (1984) we can get the review of the chaotic synchronization.

Finally in 1990, researchers realized that chaotic systems can be synchronized. The chaos synchronization has been seen as a topic of independent research with the research of Pecora and Carroll [17, 18, 19], along with the Ott, Grebogy and Yorke [20]. In the research of Pecora and Carroll [17-19], where a synchronization method was established by coupling two identical chaotic dynamical systems through transmission of a driver as a subsystem which will act as a chaotic signal which is common between them. And researchers have realized that chaotic systems can be synchronized.

A. *Preliminaries*

Synchronization of chaos is a phenomenon that may occur when two or more, chaotic oscillators are coupled, or when a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect, which causes the exponential divergence of the trajectories of two identical chaotic system started with nearly the same initial conditions, having two chaotic systems evolving in synchrony might appear quite surprising. Sensitivity to initial conditions is popularly known as the "butterfly effect", so called because of the title of a paper given by Edward Lorenz in 1972 to the *American Association for the Advancement of Science in Washington, D.C. entitled Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?* The flapping wing represents a small change in the initial condition of the system, which causes a chain of events leading to large-scale phenomena. Had the butterfly not flapped its wings, the trajectory of the system might have been vastly different.

B. *Chaotic Behavior*

The study of dynamical systems was based, for a long time, on examples of differential equations with regular solutions. If these solutions remained in a bounded region of the phase space, then they correspond to one of two types of behavior: a stable equilibrium point or a periodic (or quasi-periodic) oscillation. In 1961, working in a simplified model of atmospheric transfer with three nonlinear differential equations, he observed numerically that making very small changes in the initial conditions he got a huge effect on their solutions. It was the evidence of one of the main properties of chaotic dynamics which was later known as sensitive dependence on initial conditions. This property had already been investigated from the topological point of view by Poincare who described it in his monograph "Science and Method" (1903).

In the ancient mythology and philosophy, the word chaos meant the disordered state of unformed matter supposed to have existed before the ordered universe. The combination "Control of chaos" assumes a paradoxical sense arousing additional interest in the subject. Many deterministic nonlinear systems exhibit, more complex invariant sets which act as attractors for their dynamics inspite of fixed-point solutions and limit cycles. The experiment that boosted the consideration of chaotic behavior was due to Lorenz [3]. T. Li and J. A. Yorke were the first authors who in 1975 introduced in their paper "Period Three Implies Chaos" [2] the term "chaos" or more precisely, "deterministic chaos" which is used widely. In this paper Li and Yorke presented the study of possible periods of periodic points which was defined for continuous real maps in an interval of real numbers. A continuous map which is given by $f: I \to R$ and it would be chaotic when

a) f has periodic points of period $n, \forall n \in N$,
b) There is an uncountable set $J \subset I$ such that,

$$\lim_{n\to\infty} sup\left|f^n(x) - f^n(y)\right| > 0\, and$$

$$lim_{(n\to\infty)} inf \mid f^n(x) - f^n(y) \mid = 0 \forall x, y \in J$$

c) If x^* a periodic point of f is valid and for all

$$x \in J, \lim_{n\to\infty} sup\left|f^n(x) - f^n\left(x^*\right)\right| > 0$$

It was concluded from condition (a) and (b) that any two orbits can successively move away and approach over time in J orbits,and at any point of J the periodic points are not asymptotic respectively. A map f which has 3-periodic orbit is chaotic considered as a Schakowsky's result in a paper written in Russian [4] in 1964.This continuous maps is based on a certain natural numbers which were ordered and designated by Schakowsky's sequence.

Here is the Schakowsky's ordering-

$$3 < 5 < 7 < 9 < \ldots\ldots < 2.3 < 2.5 < 2.7 < \ldots\ldots < 2^2.3 < 2^2.5 < 2^2.7 < \ldots\ldots < 2^3.3 < 2^3.5 < 2^3.7 < \ldots\ldots < 2^4.3 < 2^4.5 < 2^4.7 < \ldots\ldots < 2^3 < 2^2 < 2 < 1 \tag{1}$$

Theorem 1. Assume that f is a continuous map on an interval and has a period p orbit. If $p < q$ then f has a period $-q$ orbit.

Because of no single formal definition of the deterministic chaos, the behavior of chaotic systems can be defined as an observable pattern that appears unpredictable and irregular in large time scales.

In common usage, "chaos" means "a state of disorder" [5]. However, in chaos theory, the term is defined more precisely. Although there is no universally accepted mathematical definition of chaos, a commonly used definition says that, for an invariant subset $E \subseteq X$ of a dynamical system $S = (X, T, \varphi^t)$ to be classified as chaotic, it must have the following properties [6]: that φ^t must be sensitive to initial conditions, φ^t must have topologically mixing in E and periodic orbits of φ^t must be dense in E. Sensitivity to initial conditions means that each point in such a system is arbitrarily closely approximated by other points with significantly different future trajectories. Thus, an arbitrarily small perturbation of the current trajectory may lead to significantly different future behavior.

However, it has been shown that the last two properties in the list above actually imply sensitivity to initial conditions [7,8] and if attention is restricted to intervals, the second property implies the other two [9] (an alternative, and in general weaker, definition of chaos uses only the first two properties in the above list [10]. It is interesting that the most practically significant condition, that of sensitivity to initial conditions, is actually redundant in the definition, being implied by two (or for intervals, one) purely topological conditions, which are therefore of greater interest to mathematicians.

In the phase space the presence of a chaotic attractor which has an infinite dense set of unstable periodic orbits insures that it is impossible to determine the position of the system in the attractor over time, even we know its position on that attractor at earlier time. As we know that in detail for hyperbolic dynamical systems, the relationship between the trajectories of the chaotic attractor and the unstable periodic orbits, for which the separation into stable and unstable invariant subspaces is consistent under the dynamics evolution [11].The fundamental role play in the un-stabilization mechanism of that attractor by infinite set of unstable periodic orbits which is in a chaotic attractor located in some symmetrical invariant manifold, since it is responsible by the dynamics of phenomenon such as riddling attraction of basin and bubbling of chaotic attractor [12,13].

The remainder of this paper is organized as follows. In the Section II, chaos synchronization and synchronization via. Coupling has been described. Synchronization of such a complex dynamical network and some coupling mechanism has been presented in Section III. Finally, some concluding remarks are provided in Section IV.

II. CHAOS SYNCHRONIZATION

Synchronization of chaos is a phenomenon that may occur when two, or more, chaotic oscillators are coupled, or when a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect, which causes the exponential divergence of the trajectories of two identical chaotic system started with nearly the same initial conditions, having two chaotic systems evolving in synchrony might appear quite surprising. However, synchronization of coupled or driven chaotic oscillators is a phenomenon well established experimentally and reasonably well understood theoretically. The dynamics of a system would be chaotic in behavior when it never repeats itself and even if initial conditions are correlated by proximity, the corresponding trajectories become uncorrelated. As such, the possibility of two (or more) chaotic systems oscillates in a synchronized way is not an obvious phenomenon, since it is not possible to reproduce exactly the same initial conditions.

As we know that it is impossible to reproduce exactly the same initial conditions and parameters, then by the effect of a sufficiently strong coupling we can change and counter balance the track of the trajectories to diverge due to chaotic dynamics. As a result, there is a possible way to reach full synchronization in chaotic systems whereas they are coupled by a suitable dissipative coupling. In order to "force" chaotic systems follow the same trajectory in the chaotic attractor, by applying small disturbances between the systems we can couple the synchronizing chaotic systems. Still Under the influence of external noise, two synchronized may lose the stability of synchronization. After a finite transient time, these trajectories will be coming again and can synchronize again due to ergodic property of chaotic trajectories.

A. Chaotic System Coupling

Although there are no various ways to couple chaotic systems, the coupling must have certain relevant properties. It is intended that the coupling (i) is dissipative, means, that tends to make the state vectors of the chaotic systems coming together, and (ii) it does not affect the chaotic state which has been synchronized. There is a possibility to consider two coupling mechanisms: unidirectional (one-way or directional) and bidirectional (mutual or global) coupling. In unidirectional coupling, only the dynamics of the response or slave system is affected by the dynamics of the drive or master system. The bidirectional coupling incorporates the mutual interaction between the systems.

The effectiveness of a coupling between these systems of equal dimension is given, by the analysis of the difference between the coordinate points of the respective variables of the systems or synchronization error. When the synchronization error converges to zero over time the coupling between chaotic systems leads to its asymptotic synchronization in optimal case. Two dynamical systems $S_1 = \left(X, T, \varphi^t\right)$ and $S_2 = \left(X, T, \varnothing^t\right)$ and will be in asymptotic synchronization if $t \to \infty$ and

$$\lim_{t\to\infty} \varphi^t(x) - \varnothing^t(x) = 0 \tag{2}$$

for a limited coupling strength between them.

And two dynamical systems $S_1 = \left(X, T, \varphi^t\right)$ and $S_2 = \left(X, T, \varnothing^t\right)$ and will be in practical synchronization if $t \to \infty$ and

$$\lim_{t\to\infty} \left\| \varphi^t(x) - \varnothing^t(x) \right\| < K \tag{3}$$

for a positive constant $K < 1$.

Whenever there is no possibility to achieve practical synchronization, but the difference between the dynamical variables of the systems is bounded, we can still apply the technique of chaos control. Note that, the dynamics of the chaotic systems introduces new freedom degrees in sets of coupled systems. However, in general freedom degrees for the coupled system actually decrease, when two (or more) chaotic oscillators are synchronized via coupling.

Hence, the effectiveness of a coupling between these systems of equal dimension is given, by the analysis of the difference between the coordinate points of the respective variables of the systems or synchronization error.

III. COUPLING MECHANISM

A. Identical Synchronization

Chaos synchronization began with the studies of *Kaneko* [15] And *Afraimovich* [16], about coupling of discrete and continuous identical systems having different initial conditions. The identical synchronization may occur straightforward when one of them drives the other or two identical chaotic oscillators are mutually coupled. If $(x_1, x_2, x_3, \ldots.., x_n)$ and $\left(x'_1, x'_2, x'_3, \ldots\ldots x'_n\right)$ denotes the set of dynamical variables of the chaotic systems, which are the states of the drive and response systems. It is said that identical synchronization occurs when there is a set of initial conditions $(x_1(0), x_2(0), x_3(0), \ldots.., x_n(0))$ and $\left(x'_1(0), x'_2(0), x'_3(0), \ldots\ldots x'_n(0)\right)$ such that, if we denote the time by t , then for $t \to \infty$

$$\left| x'_i(t) - x_i(t) \right| \to 0, \quad for \;\; i = 1, 2, 3, \ldots\ldots.., n \tag{4}$$

That means in a good approximation the dynamics of the two oscillators verifies $x'_i(t) = x_i(t)$ for $i = 1, 2, 3, \ldots, n$ when large time enough taken in to consideration. This is called the synchronized state in the sense of identical synchronization.

Many of the fundamental concepts in chaos synchronization are given by Afraimovich [16]. In the research of Pecora and Carroll [17-19], where a synchronization method was established by coupling two identical chaotic dynamical systems through transmission of a driver as a subsystem which will act as a chaotic signal which is common between them. Hence, the method of synchronization by Pecora and Carroll, also used it as complete replacement, suggests how state of a synchronous chaotic systems can be used as a driver in communication.

Thus, for a given chaotic system, such type of synchronization method requires decomposition in order to obtain an appropriate driver subsystem. Hence, in order to identify driver subsystem which is stable, they are usually tested several combinations of a subset of state variables. It appears like counter intuitive that a non-dissipative system which can leads to the synchronization, but which is in a multidimensional volume preserving chaotic dynamical system, and in order to allows choosing a stable subsystem there must be at least one contractor direction so that volumes in phase space are preserved. Given the possibility of the synchronization of chaotic systems, it is compulsory to determine conditions under which the synchronization of chaotic systems is stable. In these papers of Pecora and Carroll [16, 18] presented the first response to this question, beyond to generate the coupling mechanism, which is relatively straightforward but deceptively simple.

According to Fujisaka and Yamada [23, 24], the general way to introduce a bidirectional coupling between two identical chaotic systems by adding symmetric linear coupling terms to the expressions that define them. This type of coupling mechanism, which may be total or partial, is known as linear diffusive coupling. A study of Stefański [25] states that the properties of exponential divergence and convergence in total coupling which allow to estimate the largest Lyapunov exponent of any of the chaotic dynamical system, there is a possibility which is especially advantageous in non-smooth systems, where the Lyapunov exponents' estimation is not straightforward.

B. Generalized Synchronization

We have come to the conclusion that there are two central issues: First is that one should generalize the concept of synchronization including nonidenticity between the coupled chaotic systems. And the second is that one should design some tests to detect the nonidenticity. A lot of researchers have shown that this type of chaotic synchronization can exist [21, 22]. Mainly, this type of synchronization occurs when one has different coupled chaotic oscillators, although it has also been reported between identical oscillators. Given chaotic dynamical variables $(x_1, x_2, x_3, \ldots.., x_n)$ and $(y_1, y_2, y_3, \ldots..., y_n)$ that determine the state of the oscillators, generalized synchronization occurs when there is a functional, Φ, such that, after a transitory evolution from appropriate initial conditions, it is

$$[y_1(t), y_2(t), y_3(t), \ldots..., y_n(t)] = \Phi[x_1(t), x_2(t), x_3(t), \ldots, x_n(t)]$$

This states that the dynamical states of one are completely determined by others. If both the oscillators are mutually coupled this functional, Φ has to be invertible, if it has driven and response configuration the evolution of the response is determined by drive, and Φ does not need to be invertible. When Φ is identity, identical synchronization will be the particular case of generalized synchronization.

One knows that the response system is asymptotically stable if there is a synchronization function which transforms each and every trajectory in the attractor of the chaotic transport system into a chaotic system trajectory in the attractor of the chaotic response system. In this case, the synchronized trajectories are located in a stable synchronization manifold. Based on the equivalence of generalized chaotic synchronization in the coupled chaotic system and asymptotic stability of the response system, Abarbanel [26] established a criterion for detecting generalized chaotic system synchronization called the auxiliary system approach.

Studies have shown that the generalized synchronization includes the identical synchronization as a particular or special case, in which the functional relationship, i.e. Φ should be identity function and the synchronization manifold should have a hyper plane. However, while the identical chaotic synchronization is easily seen by representing the difference between the coordinates of the two chaotic systems in coupling, to detect generalized synchronization or generalized chaotic synchronization does not follow a simple method, especially when one can analyze information obtained via experimentally.

C. Phase Synchronization

This form of synchronization, which occurs when the osci--llators coupled are not identical, is partial in the sense that, in the synchronized state, the amplitudes of the oscillator remain unsynchronized, and only their phases evolve in synchrony. Observation of phase synchronization requires a previous definition of the phase of a chaotic oscillator. In many practical cases, it is possible to find a plane in phase space in which the projection of the trajectories of the oscillator follows a rotation around a well-defined center. If this is the case, the phase is defined by the angle, $\varphi(t)$ described by the segment joining the center of rotation and the projection of the trajectory point onto the plane. In other cases it is still possible to define a phase by means of techniques provided by the theory of signal processing, such as the Hilbert transform. In any case, if $\varphi_1(t)$ and $\varphi_2(t)$ denote the phases of the two coupled oscillators, synchronization of the phase is given by the relation $n\varphi_1(t) = m\varphi_2(t)$ with m and n whole numbers.

D. Anticipated and Lag Synchronization

In these cases the synchronized state is characterized by time interval τ such that the dynamical variables of the oscillators, $(x_1, x_2, x_3, \ldots..., x_n)$ and $(x'_1, x'_2, x'_3, \ldots\ldots x'_n)$ are related by $x'_i(t) = x_i(t+\tau)$; this means that the dynamics of one of the oscillators follows, or anticipates, the dynamics of the other. Anticipated synchronization may occur between chaotic oscillators whose dynamics is described by delay differential equations, coupled in a drive-response configuration. In this case, the response anticipates the dynamics of the drive. Lag synchronization may occur when the strength of the coupling between phase-synchronized oscillators is increased.

E. Amplitude envelop Synchronization

This is a mild form of synchronization that may appear between two weakly coupled chaotic oscillators. In this case, there is no correlation between phases or amplitudes; instead, the oscillations of the two systems develop a periodic envelope that has the same frequency in the two systems. This has the same order of magnitude than the difference between the average frequencies of oscillation of the two chaotic oscillators. Often, amplitude envelope synchronization precedes phase synchronization in the sense that when the strength of the coupling between two amplitude envelope synchronized oscillators is increased, phase synchronization develops.

Remark: All these forms of synchronization share the property of asymptotic stability. This means that once the synchronized state has been reached, the effect of a small perturbation that destroys synchronization is rapidly damped, and synchronization is recovered again. Mathematically, asymptotic stability is characterized by a positive Lyapunov exponent of the system composed of the two oscillators, which becomes negative when chaotic synchronization is achieved.

IV. CONCLUSION

Chaotic synchronization phenomenon are quite recent in the nonlinear dynamical systems theory and still continue to raise a high interest in the scientific and engineering communities. Traditionally, synchronization has been based upon periodic signals. It has now been realized that chaotic signals can also be used for synchronization. In fact, they offer more possibilities and flexibilities. Synchronization of two or more dynamical system is a fundamental phenomenon for study in science, engineering, and technology.

REFERENCES

[1] C. Huygens, "Horoloquium Oscillatorium,"Apud. F. Muguet, Paris, 1673.
[2] T. Li, and J. A. Yorke, "Period Three Implies Chaos," Am. Math. Monthly, 82, pp.985-992, 1975.
[3] E.N. Lorenz, "Deterministic non-periodic flows," J. Atmos. Science, 20, pp.130-141, 1963
[4] A.N. Sharkovkii, "Coexistence of cycles of a continuous map of a line into itself," Ukrainian Math. J., 16, pp. 61-71, 1964.
[5] K.T. Alligood, T. Sauer, and J.A. Yorke, "Chaos: an introduction to dynamical systems," Springer-Verlag, New York, 1997.
[6] Hassel blatt, Boris, and Anatole Katok, "A First Course in Dynamics: With a Panorama of Recent Developments," Cambridge University, 2003.
[7] Saber N. Elaydi, "Discrete Chaos," Chapman & Hall, CRC Press, 1999, pp.117.
[8] William F. Basener, "Topology and its applications," Wiley, 2006, pp.42.
[9] Michel Vellekoop, and Raoul Berglund, ""On Intervals, Transitivity = Chaos," The American Mathematical Monthly, Vol.101, pp.353–355, April 1994.
[10] Alfredo Medio, and Marji Lines, "Nonlinear Dynamics: A Primer," Cambridge University Press, 2001, pp.165.
[11] J. P. Eckmann, and D. Ruelle, "Ergodic theory of chaos and strange attractors," Rev. Mod. Physics, 57, pp.617-656, 1985.
[12] Y. Nagai, and Y. C. Lai, "Periodic-orbit theory of the blowout bifurcation," Phys. Review, *E* 56, pp. 4031-4041, 1997.
[13] Y. C. Lai, "Symmetry-breaking bifurcation with on-off intermittency in chaotic dynamical systems," Phys. Review, *E* 53, pp. 4267-4270, 1996.
[14] R. M. May, "Simple mathematical models with very complicated dynamics," Nature, 261, pp. 459- 467, 1976.
[15] K. Kaneko, "Lyapunov analysis and information flow in coupled map lattices," Physica D 23, pp.436, 1986.
[16] V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, "Stochastic synchronization of oscillations in dissipative systems," Radiophys. Quantum Electron, 29 (9), pp.795-803, 1986.
[17] L. M. Pecora, and T. L. Carroll, "Synchronization in chaotic systems," Phys. Rev. Lett., 64 (8), pp.821-824, 1990.
[18] T.L. Carroll, and L.M. Pecora, "Synchronizing chaotic circuits," IEEE Trans. Circuits Systems, 38 (4), pp.453-456, 1991.
[19] L.M. Pecora, and T.L. Carroll, "Driving systems with chaotic signals," Phys. Review, A 44 (4), pp.2374-2383, 1991.
[20] Ott, C. Grebogy, and J.A. Yorke, "Controlling chaos," Phys. Rev. Lett., 64 (11), pp.1196-1199, 1990.
[21] V.S. Afraimovich, N.N. Verichev, and M.I. Rabinovich, "Izvestiya vysshikh uchebnykh zavedenii," Radio Physica, 29 (9), pp.1050, 1986.
[22] N.F. Rulkov, M.M. Sushchik, and L.S. Tsimring, "H.D.I. Abarbanel," Phys. Review, E 51, pp. 980,1995.
[23] T. Yamada, and H. Fujisaka, "Stability theory of synchronized motion in coupled-oscillator systems," Prog. Theoret. Physics," 69 (1), pp.32-47, 1983.
[24] T. Yamada and H. Fujisaka, "Stability theory of synchronized motion in coupled-oscillator systems II," Prog. Theoret. Physics, 70 (5), pp.1240-1248, 1983.
[25] A. Stefański, "Estimation of the largest Lyapunov exponent in systems with impacts," Chaos, Solitons and Fractals, 11, pp.2443-2451, 2000.
[26] H.D.I. Abarbanel, N.F. Rulkov, and M.M. Sushchik, "Generalized synchronization of chaos: The auxiliary system approach," Phys. Review, E 53, pp.4528-4535, 1996.