International Journal of Scientific Engineering and Technology (ISSN : 2277-1581)
www.ijset.com, Volume No.1, Issue No.4, pg : 28-34 01 Oct. 2012

Philosophical Ramblings on Some Mathematical Dichotomies and the Concept of Scale

Sundaram Ramchandran
SUNY Binghamton, Binghamton, NY, Aug 2001
Email: sundarramchandran@hotmail.com, zenkalidas@yahoo.com

Abstract

This article looks at the issues relating to and arising out of self similarity of scale with links to "fractal" geometry and more generally investigates the possibility of finding the limit of a dynamical process given an initial generator / set of generators as also the inverse problem of finding the initial image that results in a particular attractor given the process. The latter could also be related to finding the point at which the image vanishes (including the possibility of oscillatory phenomenon).This problem is viewed through the "Operator paradigm" and links to recursive reflective phenomenon such as consciousness are examined. The above is also linked to the "Satisfiability problem" in computer science with links to Godel's theorem in mathematical logic. At a more constructive level, methods of constructing more general shapes out of the basic shapes such as "Seiperenski's Triangle" through basic operations (including diffraction) are examined with the possibility of lifting the results obtained to functions defined on the above shapes. Some paradoxes relating to scale which emerge in this process such as Zeno's paradox and other related paradoxes are examined. The article concludes by examining the possibilities of applying the outlined techniques to mathematical logic / meta-mathematics itself underlining the constructive nature of mathematical proof (with possible links to the area of optimization, especially discrete and combinatorial optimization).

Keywords : Self-similarity, Fractal Dimension, Complex Dimension, unit scale, reciprocal / frequency space, Solitons, Conceptual lattices, Sieperenski's Triangle, Attractor/Limit, Limit Cycle, Iterated Function Systems, Operator, Zeno's Paradox, Infinite Series, Chebyshev Polynomials,Godel's theorem, Heisenberg's Uncertainty Principle, Truth Value.

INTRODUCTION

In mathematics and in science in general, phenomenon occur at various scales ranging from cosmic (as in astrophysics) to sub-atomic (as in the case of nano-technology, nuclear physics and quantum chromodynamics). To this we can also add "laws" governing phenomenon from other fields

As usual, humankind has exploited symmetry across scale both as a conceptual paradigm and as a simplification strategy. For example, the concentric sphere model occurs at various models ranging from the solar system through the structure of atoms and molecules to the shell model of the nucleus.

Of course, there is "emergent" phenomenon in the sense that concepts emerge whose "significance" are more than that of the sum total of their basic ideas. But it is right that it should be so, ontologically since, otherwise all biology can be reduced to chemistry and all chemistry to physics and all physics to mathematics and most other disciplines (Including neuro-sciences, history, aesthetics etc) to mathematics.

One could look at the above as "solitons" in the space of concepts from various disciplines emerging as major "invariant" and "significant" entities out of the dynamics of the learning across those disciplines. The "emergent" phenomenon then could possibly be explained as "higher order" solitons emerging out of the interaction of lower order solitons.

Of course, in a philosophical sense, mathematics comes close to consciousness studies in the sense of examining what is "significant" from a human perspective which not only helps to add flesh and sinew to a discipline but also adds to the basic framework of the subject as well as to the "mind" of the subject.

In the course of practicing mathematics in this sense, various dichotomies emerge such as:

- The finite vs. infinite
- The countable vs. uncountable (integers vs. real numbers)
- The curved vs. the straight
- The rough vs. the smooth

And so on…

One major theme common too many dichotomies are that of scale. For example, the whole concept of "atom" is attributed to the Greek scientist, mathematician and philosopher Democritus who seems to have conceived it as that which remains invariant after a process of splitting. (Mathematicians would possibly conceive of the same as a fixed point of the "splitting" operator / mapping). Of course, later developments have involved entities and concepts at a high resolution than the above (such as that of sub-atomic particles such as quarks etc).

One major theme that combines the theme of symmetry and scale as well as that of the rough / smooth dichotomy is that of "fractals' which are said to be self-similar across scale and which are supposed to exhibit a dependence of the various measures as a function of the form :
Constant * (dimension)^exponent where the generalized exponent could be fractal.

Let us examine these issues by looking at the fractal called Sieperenski's triangle:

Some major questions that can be asked:

- The fractal dimension of the set (Hausdorff dimension for example)

A more interesting and general question relates to asking what would be the limiting shape of a recursive process similar to Sieperenski's and with various generators (initial shape) (In the case of Sieperenski's, it is dark triangle) and conversely, what shape to start off with to realize a given limiting shape / distribution. Incidentally, this motif may possibly be related to the Sree Yantra in the iconography, mythology, symbology etc of the Hindu Religion.

Note: What makes the above problem interesting is that the connectivity of the motifs matters. One interesting problem given the central placement of the holes in a statistical sense relates to the statistical proportion of the holes which mark a turning point (for example propagation of a wave across resistive material with links to superconductivity etc)

Aside :

The concepts outlined above are linked to the idea of sequences and series of numbers, extended to functions and other entities (especially infinite series)

One related Paradox is the Zeno's paradox. Some thoughts on the same:

- If the relative speeds were known before hand, then it becomes a simple algebraic exercise (of course it is unrealistic)

- If back tracking is allowed, then it could possibly lead to (possibly damped) oscillations (links to phase locked loops??) such as those in a pendulum. (A simpler version of course is related to methods of summation of infinite series but this of course leads to the issue of the difficulty of infinite steps but of course one could also look at the above occurring at infinite instants of time)

Of course a basic unit of length (for example Planck length) would be useful not only in bounding the number of steps but also in tracing irregular curves. Of course the latter would need an appropriate shape (possibly irregular) in terms of which the above could be measured (links to the concepts of rectification of curves)

The above is also related to the paradoxes that emerge when scaling down below unit length. If we choose an epsilon ep that is below unit length 1, one immediate paradox relates to the fact that ep^3 (and generally ep^n with n greater than 1) << ep. This leads directly to the paradoxical fact that the volume is less than the length of a side. This possibly necessitates defining an area / volume etc as increasing functions of the basic units.

- One interesting point is that though ep^3….ep^n… are increasing functions of ep, ep^2 …ep^n < ep when ep < 1. In some senses, this relates to the sub-space idea that when we are looking at scales below the unit length, events at scale x are independent of events at scale x^2 (the idea of linear independence etc). Though this could be resolved, it would be interesting to investigate the implications towards theorems such as Fermat's that possibly rely on such order of magnitude comparisons between lengths, area …. (with the possibility of Fermat's theorem not holding in general for variables whose magnitude is less than 1.

- One could possibly argue that it does not make sense to talk about a unit length other than 1, thus making ep the new unit of measure. Essentially, it involves rescaling / dividing by ep which would actually scale up all units by a factor of 1/ep with higher dimensional measures suitably scaled up. This also relates to the idea that essentially a unit less than 1 only makes sense when comparing 2 measures., in which case it does make sense to state : if x is > 1, then not only is 1/x not only < x and < 1 but (1/x)^n < (1/x) < 1 < x

This also seems to relate to the crucial idea that 1/ep represents a unit in reciprocal / frequency space. Philosophically speaking, this seems to make sense that to measure any quantity below unit scale, one would have to use imaging techniques using EM / acoustic waves thus involving both the time(space) and frequency planes (links to so called sub-wavelength imaging etc). This would relate to density distributions defined on the above sets / volumes / areas etc even more markedly. This directly seems to relate to the idea that functions and operations on them are in a sense more basic than numbers thus leading to ideas such as harmonic decomposition etc. Even more philosophically, this seems to lead to Penrose's ideas of the universe being built on a substrate even more basic than that of space and time (twistor space etc).

These ideas directly seem to relate to quantum mechanics as well as fractal geometry.

Another possible resolution could relate to the fact that most of the "measure" is essentially concentrated in the arc length (which could be almost infinite or "unrectifiable" in mathematical terms) while all the higher exponents are zero (links to the idea of "measure zero" sets of uncountable cardinality) (links to Mobius strip etc ??). One pictorial idea is that in directions other than the required direction / set of directions, the measure is essentially zero (links to finding axis of inertia etc). This also relates to the fact that a measure of a closed set includes the measure enclosed in the interior as well as the measure concentrated on the boundary. Could this be one possible method of constructing systems with ray like projections/ rays emanating in an infinite number of directions but essentially not enclosing a region and / or not enclosing a boundary (links to dynamical systems with no limit cycles etc?). The above directly relates to the ideas of fractal measure in the sense of volume / area / generalized measured (as per the exponent covered by a set) can be related to the average measure in a sense (relative to the unit ball / square etc)

One interesting application of the above in conjunction with the ideas of self-dual functions mentioned later in the document leads to the idea of sharpening / localizing signals (in physical / phase space) through a kind of "fractional convolution" of the signals / functions

An interesting idea is to cast the problems in frequency space or a combined time / frequency analysis. This also seems to be linked to the oscillations / phase locked loops idea. This also makes sense due to the oscillatory mechanisms that are at the heart of many biological processes including locomotion (and vision which seems to be at the heart of estimating speed, distance etc possibly through inbuilt trigonometric means)

A related idea is that of a lower bound on the combined precision in the time frequency plane with links to the Heisenberg uncertainty principle that intuitively states that the product of the uncertainty in frequency and that in time cannot be less than a certain constant. One idea seems to be that of constructing functions known to be self-dual (equal to its own Fourier transform) and thus bypassing the inequality

Another resolution of the paradox related to

The above also seems related to the fact that to the outsider, the steps of Achilles (who is tracking the tortoise) may seem random as in Brownian motion etc. The above also seems related to the idea of tracking a curve (such as the pursuit curve) without knowing the closed form equation by proceeding along the tangents.

The above problem relating to the Sieperenski's triangle can be considered a basic building block since other shapes can be triangulated (including curvilinear shapes such as that on curved manifolds such as spheres) as also many other images including color can be built up of operations such as diffraction (looking at the triangle as a mask), convolution, addition and related operations. Related questions include that of the point of view / focus etc.

METHODS

A Mathematical approach to the above

Motivation

Building up other shapes based on the triangle. For example if we denote the generator and a snapshot of the triangle at a particular instant n as F(x, y), one could pose the following question:

Just as convolving the triangle with itself leads to a square, would this result in convolving F(x, y) with itself defined on a square?

Essentially, this seems to decomposing a function into step functions (both in terms of the basic substrate shape as well as that of the intensity function). This approach also seems to be linked to the Cantor set.

Of course, one could also pose the problem in terms of area preserving maps such as the bakers' map in Dynamical systems with links to the Hamiltonian etc.

Another interesting idea is to link color versions to result of diffractions with the above fractal image as mask (links to Fourier transform etc)

Another theme relates to looking at the above tiling in terms of zeros of the polynomial z^2 + z + 1 = 0 (leading to the roots being (-1 +/- i*sqrt (3))/2 with links to Eisenstein lattice etc. Essentially one label a region in one of 2 colors, depending on how points in to which root the points in that region converge using the Newtonian method. A more general idea relates to the shape of these basins of attraction (here 2) which when extended to polynomials of degree greater than 3 result in enormously rich areas of mathematics including complex analysis, fractal geometry ((as well as graphics) (with possible links to the 4 color problem as well). Another interesting example of such a function leading to rich dynamics could be chebyshev polynomials relating to fractional values of the argument such as cos (z/n) in terms of cos (z) going on to irrational and real (and possibly complex) values of a (including the golden mean) where f (z) = cos (a*z) is a chebyshev polynomial expressed in terms of cos (z). The above theme, that of chebyshev polynomial of an irrational argument / order seems to be a practical non trivial example of an infinite series with a known closed form solution with possible links to Zeno etc (along with Bessel functions, Gaussian primes etc.). Furthermore, when linked to orthogonal polynomial expansions etc, the above seems to lead to the possibility of realizing uncountable numbers from countable ones with possible extensions to arbitrary algebraic structures. From a philosophical viewpoint, the above themes seem to be related to the so called "dream" state that possibly represents jumps between various stable / mental states but which can be collapsed when faced with an emergency. One interesting possibility is that of obtaining quasi-crystals such as Penrose tilings through the above paradigm.

Other themes relate to the topics of doubly periodic functions in the plane, modular forms and automorphic functions that are invariant under the action of operators denoted by 2*2 matrices in the complex plane of which a subset of the operators/matrices consist of 0, 1, i and 1 + i and out of which the more general operators can be built.

One could cast the problem in terms of looking at the required image as a limit of sequence of images / functions / space-filling curves etc and look at a suitable distance / metric in the above space (such as that of hausdorff etc) and look at a suitable generator or motif / operator combination that would reduce the distance between the function at the nth instance and the limiting function (which could also be the fixed point / eigenfunction of the operator) (a contraction mapping in mathematical parlance)

The forward problem (possibly easier) would involve computing the limit of the above sequence of operators given an initial motif and the inverse problem would entail reconstructing the operator given the required image.

Forward Problem:

Determine the limit image g(x, y)

Lim A^n f0(x,y) = g(x,y)
N->infinity

Or determine the s within which

A^sf0(x, y) approximately equal to g(x, y) (within a certain degree of error and where the s would depend on the error) where f0(x, y) is the initial image.

In general, in the above and what follows, the exponent s could be integer valued, real or even complex
`

Inverse problem

Determine the starting image f0(x, y) and /or A for which

A^sf0(x, y) = g(x, y)

One could expand both f0(x, y) and g(x, y) in terms of the basis functions of the space of images

Fixed point problem

Find the s for which A^sf0(x, y) = f0(x, y) (links to eigenvectors etc)

If one wants a fixed point solution that does not oscillate (not a cycle), we would need A to be a contraction mapping,

i.e.

d(Af(x,y),g(x,y)) < Cd(f(x,y), g(x,y))

Where |C| << 1

One could also choose the operator A to be a well known operator related to differential equations such as diffusion / heat equation etc or the evolution operator.

One interesting case arises when g(x, y) = 0 or 1 or a certain constant (real or complex).
If A is not a contraction mapping, one could find a set of scales to which different sets of the image resonate and where the behavior of g(x,y) oscillates in scale (links to so called “complex dimensions” or “discrete scale invariance” or “periodicity in scale space” as also the idea of muti-fractals). This seems to have analogies to the propagation of a solitary wave that retains its waveform and whose beam width expands and contracts as it propagates. It would also be interesting to link the period s to the scale.

SOME INTERESTING RESULTS

Another interesting idea is to look at the triangles / lattices as limits of sequence of continuous curves such as Fermat Curves as also the reverse (continuous curves as the limits of lattices at various stages)

One conceptual leap would be to look at the operators / the associated matrices themselves as images (triangular matrices) which can be extended / lifted to the operators acting on themselves in a self-referential manner (links to Godel, coding theory etc). Of course the above has to have a mathematical framework. Another interesting idea is that of linking the space of operators to the space of graphs (including infinite dimensional extensions) and lifting the relationships among graphs/operators (algebras) to the functions associated with and induced on these algebras. An interesting application could be in the area of expanding graphs with possible links to mathematical biology (pathway analysis etc) as well as areas related to belief systems etc. A related idea is the concept of “transitive closure “of a graph. Again revisiting Godel, one could relate the Sieperenski’s triangle to assigning truth values to propositions in the complex plane. Hence a trajectory in the space of continuous functions over the complex plane could represent a possible proof of a mathematical proposition.

Again, mathematically if we look at the space of images which can be considered the space of functions from C (the complex plane) to itself, we could look at evolution operators which realize geodesics in this space while satisfying constraints and avoiding singularities / defects etc (possible links to homotopy theory and possibly deformation algebras etc)

Once the above machinery is set up, one could try to accelerate convergence by using methods such as Euler Shanks methods that are used in the case of sequence /; series of numbers to sequences of functions etc (This could also be related to the Zeno's Paradox)

Extensions would involve looking at solving differential equations with the above irregular shapes as boundaries / curves or more practically substrates including the possibility of the above emerging as solutions to differential equations (links to quasi-crystals etc)

The above themes seem to be linked to Gödel's' Theorem in the sense that a proposition is not true or false but can be assigned a value which can be thought of as an amplitude of a wave and the truth values can be said to emerge from the interaction of the various propositions which can be said to be similar to path integrals representing interference between various paths (where each path is treated as a wave). This is somewhat different from the probability assigned to path and can be said to be similar to ideas from belief systems / possibility theory etc where beliefs that are in harmony with other beliefs and also with the reality as represented by known facts have a greater weight than beliefs which are not in harmony. This could also be related to ideas such as constructive and destructive interference as also negative weights representing disbelief in propositions. The above also seems to be linked to Quantum Logic. As an aside, one interesting possibility is that of links between the Sudoku puzzle and quantum, logic in the sense that the relationships between various constraints and the constraints themselves can be used to solve the problem without necessarily using trial and error (of course, depending on the number of constraints vs. the number of variables thus leading to the themes of rank etc)

This directly relates to realizing a mapping / coding from the space of functions representing a proposition to a real number between 0 and 1 or between – 1 and 1 (including possibilities such a function that is zero over all rational numbers and 1 (or any other value) over all irrational numbers or vice versa.

An interesting idea in this context is to represent the syntax of a proposition by the coefficients of a polynomial and the semantics / values for which the proposition is true by the factors / roots of the polynomial.

In a sense, checking if a proposition is well formed (in the sense of compilation) can be said to be a syntactical affair while proving it / realizing its meaning can be said to be a semantic issue.

So if we relate the proof of a theorem to location of zeros, in the complex plane, all propositions become decidable (by the fundamental theorem of algebra) with a possible proof being related to a set of derivations which can be thought of as a factorization of the associated polynomial. Of course, there may be prime / irreducible polynomials if we restrict ourselves to sub-sets of the complex plane (lattices of pairs of Gaussian Primes etc) which would translate themselves to undecidable propositions in certain models of the universe.

I wonder if this could also be the basis for realizing a mapping between different problems (with conformality/ or other properties) (Links to isomorphism / homomorphisms between graphs and functions on them). Moving from undecidability to complexity, it would be interesting to investigate if this could form the basis for an NP-P transformation. One interesting idea relates to the possibility of applying Euler Shanks type methods to reduce complexity of algorithms (and possibly of proofs)

CONCLUSION AND POSSIBILITIES (Essentially related to Mathematization of Logic and the constructive nature of a Mathematical Proof)

One could possibly extend these ideas to more abstract spaces. For example, we could look at a space of concepts if we look at each concept as being represented by an ordered or unordered collection / set of images / signals. Here we could look at natural language statements as being defined over the above space of concepts. In fact, we could look at these statements as paths / curves in the above space of concepts. An interesting extension of the above is to the area of belief systems.

One could look at logical processing of the above statements as a mapping from the above space of concepts to [0, 1] in R. Of course, this involves an assumption of fuzziness which seems "real"istic!!

One could possibly be more ambitious if we restrict ourselves to mathematical statements and propositions. One interesting idea that generalizes Godel's mapping of mathematical propositions to numbers is to map each proposition to a point in the complex plane. Some themes related to the same:

- Looking at a mathematical proposition as a link between different mathematical concepts. An imaginative way of conceptualizing the above could be to look at the propositions as curves or more generally relationships / equations between concepts which can be projected down to the curves in the complex plane. This also makes sense in the sense of embedding of conceptual graphs in the complex plane. Of course, in general, this would be multidimensional (R^n instead of just R2). Proofs would essentially be similar to lifting the above to a higher level, i.e. looking at a set of propositions / beliefs, i.e. a set / family of curves. One could also possibly look at Profs / derivations as equations/curves in the space of operators. Of course, one could also look at the above as mappings from the above space to the binary variable , 0 or 1 and more generally, the real variable [0,1] which could also be extended to fitness measures other than true / false in the strict logical sense , for example internal harmony / consistency, harmony with other belief systems, comprehensiveness, simplicity, applicability etc. The above could also represent probability / possibility etc. This seems to be borne out by the link between one set of beliefs subsuming others and the idea of one set containing other sets.

- This also makes sense if we look at the above as measurement of a state of a system (for example variables such as spin). This seems directly related to the dichotomy between the ordinal quantum numbers labeling the states which are discrete and the continuous eigenfunctions (wavefunctions ?) as well as the amplitudes that describe the state. If one look at a class as a cluster (of concepts for example), one could describe the cloud / wave-function shaped cluster in terms of the eigenfunctions. This also seems to be linked to the dynamics of spin as also to the idea related to path integrals that the relationship between fuzzy sets / wavefunctions (possibly fractal) could lead to crisp choices which seems to be connected to the raman-bragg transition as q increases leading to the classical ray picture in optics.

- The other approach of course is to look at a proposition as a polynomial in the appropriate space with the AND operator / Intersection operator denoting / denoted by multiplication and the OR / UNION operator denoting / denoted by addition. The proof essentially would involve a kind of factorization and the extraction of the roots (with links to the derivation). This also seems to be related to the syntax / coefficient space and the space of roots / factors (which seems to be a Fourier transform pair). Another interesting idea is to link the transform pair to transformations between sums and product form of propositions and between the related sums of products and product of sums

- Another idea relates to linking the above to the joint localization in time-frequency space and relating the logical questions to the above localization and the limits relating to localization (8with links to Heisenberg's uncertainty principle etc) to the problems of undecidability. Godel's incompleteness, Turing completeness, satisfiability problems in computer science etc. Some themes in this context :

  - Relating the problem of localization to the joint truth values of a proposition and its converse (for example p->q q->p, necessary and sufficient condition, iff etc). Initially, the complement of a proposition seems a good idea but it is not really a conjugate or a transform pair. Of course it might lead to there being evidence for both a proposition and its negation (p + not p >= 1??) which would possibly be a realistic assumption.

  Of course, one possibility of joint localization in the time frequency plane is the use of overlapping orthogonal functions in the time frequency plane (possibly more than 2) (intersection of corresponding areas). The other idea of course is to construct functions that are self-dual and hence automatically localized. The latter possibly directly relates to Gödel's' proof by constructing a proposition about a number and which is coded by the same number and which is possibly asserted by the proposition in a self-referential manner (of course it has to have other uses too with possible links to the genetic code, cryptography etc).

The above approaches seem to relate the space of operators/proofs lifting the functions to a higher level and the operator itself being represented by a number / function in the complex plane. Of course, in the complex plane, by the fundamental theorem of algebra, an nth degree polynomial has n roots but one can restrict oneself to subsets of the plane (Gaussian integers etc)

There also seem to be links to fixed point theorems and the associated idea of eigenvectors / eigencurves of an operator as well as a kind of consciousness in the sense of introspection etc.

This also seems to be related to the idea that proofs like criticisms can be creative/aesthetic in their own way and add value to the body of knowledge

The above could have possible applications not only in computer aided theorem proving etc as well as satisfiability problems in computer science with possible links to the mathematical proofs of correctness of software programs. It would be interesting to extend the concept of limits of sequences of functions, fixed point theory etc to prove that a particular program/algorithm will converge to the right answer (halting problem etc). More interestingly, it would be helpful in determining / constructing models of the universe and contexts in which certain sets of propositions / beliefs (especially if they have "value" in a sense) would hold true as also in constructing "useful" and interesting propositions that are tautologies (links to Smullyan's puzzles) which can also be considered mathematical invariants. The idea of "contexts" / models seems related to the idea of Galois connections in concept lattices linking concepts and objects and related to the idea of subsethood and a hence kind of ordering with links to the contraction principle and the related fixed point theorem. This also seems related to the idea that just as a drug that is not suitable for a particular disease across all sections of the population may be suitable for a certain sub-set of the population , a proposition that is not generally true may be true (or have a high probability / possibility of being true) under a set of contexts.

Interestingly, the topics mentioned above, instead of converting mathematics to logic extend the principle of self-referentialism in the sense of applying mathematics to meta-mathematics itself.

REFERENCES

1) Chaotic Logic: Language, Thought, and Reality from the Perspective of Complex Systems Science (IFSR International Series on Systems Science and Engineering] Ben Goertzel "Godel,escher, bach, an eternal golden braid" - Douglas Hofstadter

2) "Metamagical themas" - Douglas Hofstadter Nonlinear Dynamics of a Bragg cell under intensity feedback in the near-Bragg, four-order regime".Applied Optics / Vol 41, No 29, / October 2002 - S.Ramchandran and M.Chatterjee,

3) "Feedback correction of angular error in grating readout", SPIE Conference, San Diego, July 2001, vol. 4470, pp. 127-137 *- M.Chatterjee and S.Ramchandran,