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On New Quadrature Rules Based On Mixed Spline Functions

Hamsapriye

Department of Mathematics, R. V. College of Engineering, Mysore Road, Bangalore - 560059. (INDIA)
Email : hpriye@gmail.com

Abstract

The classical theory of cubic splines and the ideas of mixed interpolation are combined for developing new quadrature rules. A class of, what can be called as, mixed cubic splines are derived as an extension of the classical cubic splines which can approximate oscillatory functions $f(x)$, by a function $\tilde{S}_{\Delta}(x)$ of the form $\tilde{S}_{\Delta}(x) = a'\cos(kx) + b'\sin(kx) + c'x + d$, on a partition of subintervals of the interval $[a,b]$. This function has a property that it is twice continuously differentiable at the endpoints of the subintervals. The idea is that $f(x)$ is approximated by n such $\tilde{S}_{\Delta}(x)$, where n denotes the number of subintervals partitioning $[a,b]$. A minimum norm property is established by allowing discontinuity in $\tilde{S}''_{\Delta}(x)$ at the nodal points. Also, quadrature rules are derived based on this approximation and a few examples are considered for numerical purposes.

Keywords: Cubic splines, mixed cubic splines, Minimum norm property, Quadrature

AMS Classification: 65D05, 65D07, 65D32

1. Introduction

The theory of cubic splines is deep rooted and well established. Also the theory of mixed interpolation is well known. We can consider spline functions to be piecewise smooth polynomial curves, interpolating a given function $f(x)$ at some nodal points. Refer to Schumaker [3], Stoer [4], Sard [2] and Jones [1]. In other words, splines are locally piecewise polynomials, but are globally smooth. They are used in modeling arbitrary functions, and are used extensively in computer graphics.

Define

$$\left.\begin{aligned} X &= \{x_i : i = 0, 1, \cdots, n\} \\ Y &= \{y_i = f(x_i) : i = 0, 1, \cdots, n\} \end{aligned}\right\} \tag{1}$$

where

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}, \tag{2}$$

is a partition of the interval $[a,b]$. A given function $f(x)$ can be approximated by cubic polynomials of the form

$$S_{\Delta}(Y,x) = a\,x^3 + b\,x^2 + c\,x + d \tag{3}$$

for $x \in [x_j, x_{j+1}]$. The constants a, b, c and d are determined using the conditions that the function is twice differentiable at the points $x_i : i = 0, 1, \cdots, n$ and that $S_{\Delta}(Y, x_i) = y_i$.

The idea of approximating a given function $f(x)$ by cubic splines is based on the mixed interpolation approach of Meyer [5] and Chakrabarti [6]. The generalization of the classical cubic spline approximation by way of introducing trigonometric functions $\cos(kx)$ and $\sin(kx)$, along with a linear polynomial, where k is a free parameter, is expected to take care of the oscillatory behavior of the function. The underlying ideas are based on the oscillatory theorems of ordinary differential equations.

Given a partition P of subintervals of the interval $[a,b]$, the mixed cubic spline is a piecewise approximation of a function $f(x)$, by a function of the form

$$\tilde{S}_{\Delta}(Y,x) = a'\cos(kx) + b'\sin(kx) + c'x + d' \tag{4}$$

for $x \in [x_j, x_{j+1}]$ and k is any free parameter. That is, in each subinterval $\Delta = [x_j, x_{j+1}]$, $f(x)$ is approximated by a function as given by the relation (4), along with the conditions that $\widetilde{S}_{\Delta}(Y,x)$ is twice continuously differentiable at the nodes x_i.

In section 2, the expression for the mixed cubic spline is derived. Section 3 deals with the error analysis, by way of proving a minimum norm-type property, similar to that of the classical case, under certain restrictions. The uniqueness of the mixed spline is proved in section 4. Quadrature rules based on the new approximation are dealt with in section 5 and numerical discussions are incorporated in section 6.

2. Mixed Splines

In this section we discuss about the derivation aspects of the mixed spline approximation. The derivation is similar to that of classical splines, as explained in [4].

Define the $(n+1)$ moments M_j as

$$M_j := k^2\widetilde{S}_{\Delta}(Y,x_j)+\widetilde{S}_{\Delta}^{''}(Y,x_j) \text{ for } x_j \in X, \tag{5}$$

where k is a free parameter. It is assumed that

$$\widetilde{S}_{\Delta}(Y,x_j)=y_j;\ j=0,1,\cdots,n \tag{6}$$

Define the mesh size of the partition h_{j+1} as $h_{j+1}=x_{j+1}-x_j$.

The expression for the *mixed splines* can be derived using the following conditions:

(i) $M_j=\widetilde{S}_{\Delta}^{''}(Y,x_j)+k^2\widetilde{S}_{\Delta}(Y,x_j)$ is a linear function on any subinterval Δ;

(ii) $\widetilde{S}_{\Delta}(Y,x_j)=y_j$;

(iii) $\widetilde{S}_{\Delta}(Y,x), \widetilde{S}'_{\Delta}(Y,x), \widetilde{S}''_{\Delta}(Y,x)$ are continuous functions at the mesh points x_j.

By virtue of condition (i) and the relation (5) one can write

$$\widetilde{S}_{\Delta}^{''}(Y,x)+k^2\widetilde{S}_{\Delta}(Y,x)=M_j\frac{x_{j+1}-x}{h_{j+1}}+M_{j+1}\frac{x-x_j}{h_j},$$

for $x\in\Delta$. (7)

Solving the above differential equation, the following expression is obtained:

$$\widetilde{S}_{\Delta}(Y,x)=A_j\cos(kx)+B_j\sin(kx)+\frac{1}{k^2}\left[M_j\frac{x_{j+1}-x}{h_{j+1}}+M_{j+1}\frac{x-x_j}{h_j}\right]. \tag{8}$$

Now using the conditions

$$\left.\begin{aligned}\widetilde{S}_{\Delta}(Y,x_j)&=y_j\\ \widetilde{S}_{\Delta}(Y,x_{j+1})&=y_{j+1}\end{aligned}\right\} \tag{9}$$

one can solve for A_j, B_j and obtain

$$A_j=\frac{\left(\frac{M_{j+1}}{k^2}-y_{j+1}\right)\sin(kx_j)}{\sin(kh_{j+1})}-\frac{\left(\frac{M_j}{k^2}-y_j\right)\sin(kx_{j+1})}{\sin(kh_{j+1})} \tag{10}$$

$$B_j=\frac{\left(\frac{M_j}{k^2}-y_j\right)\cos(kx_{j+1})}{\sin(kh_{j+1})}-\frac{\left(\frac{M_{j+1}}{k^2}-y_{j+1}\right)\cos(kx_j)}{\sin(kh_{j+1})} \tag{11}$$

Thus, for $x\in[x_j,x_{j+1}]$ the expression for $\widetilde{S}_{\Delta}(Y,x)$ is derived in the form

$$\widetilde{S}_{\Delta}(Y,x)=\left(y_{j+1}-\frac{M_{j+1}}{k^2}\right)\frac{\sin(kx-kx_j)}{\sin(kh_{j+1})}+\left(y_j-\frac{M_j}{k^2}\right)\frac{\sin(kx_{j+1}-kx)}{\sin(kh_{j+1})}+\frac{x-x_j}{k^2h_{j+1}}M_{j+1}+\frac{x_{j+1}-x}{k^2h_{j+1}}M_j \tag{12}$$

It is clear from the relation (12), that the mixed cubic spline $\widetilde{S}_{\Delta}(Y,x)$ is characterized by the moments M_j. Determining these moments M_j, would determine $\widetilde{S}_{\Delta}(Y,x)$. Thus, using the conditions:

$$\left.\begin{aligned}\widetilde{S}'_{\Delta}(Y,x_j^-)&=\widetilde{S}'_{\Delta}(Y,x_j^+)\\ \widetilde{S}''_{\Delta}(Y,x_j^-)&=\widetilde{S}''_{\Delta}(Y,x_j^+)\end{aligned}\right\} \tag{13}$$

the following tridiagonal system of $(n-1)$ linear equations is obtained which has to be solved for the $(n+1)$ unknown moments M_j, which depend on k:

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$$\alpha_j M_{j-1} + \beta_j M_j + \gamma_j M_{j+1} = k^2 \delta_j, \quad j = 1,2,\cdots,n \tag{14}$$

where the coefficients are derived to be

$$\alpha_j = \frac{1}{kh_j} - \frac{1}{\sin(kh_j)} \tag{15}$$

$$\beta_j = \frac{\cos(kh_{j+1})}{\sin(kh_{j+1})} + \frac{\cos(kh_j)}{\sin(kh_j)} - \frac{1}{kh_{j+1}} - \frac{1}{kh_j} \tag{16}$$

$$\gamma_j = \frac{1}{kh_{j+1}} - \frac{1}{\sin(kh_{j+1})} \tag{17}$$

$$\delta_j = \left(\frac{\cos(kh_{j+1})}{\sin(kh_{j+1})} + \frac{\cos(kh_j)}{\sin(kh_j)}\right) y_j - \frac{y_{j+1}}{\sin(kh_{j+1})} - \frac{y_{j-1}}{\sin(kh_j)} \tag{18}$$

We notice that, in order to obtain a unique mixed spline, two more conditions are required, which are discussed in the next section.

The notion of approximating a function $f(x)$ by the mixed cubic splines can be further generalized by $f_3(x)$, of the form $f_3(x) = a'U_1(kx) + b'U_2(kx) + c'x + d'$, which is based on the well known "oscillatory theory" of ordinary differential equations (ref [7] and [8]). In this case the moments M_j are defined as:

$$M_j := \left(\frac{\overline{U}_1(kx)}{\overline{U}_2(kx)} \frac{d^2}{dx^2} - k \frac{\overline{U}'_1(kx)}{\overline{U}_2(kx)} \frac{d}{dx} + k^2\right) \tilde{S}_\Delta(Y,x)\Bigg|_{x=x_j} \tag{19}$$

where, the 'prime' in the expression (19) denotes differentiation with respect to the argument
and that

$$\overline{U}_n(kx) = U_2^{(n)}(kx)\, U_1^{(n-1)}(kx) - U_1^{(n)}(kx)\, U_2^{(n-1)}(kx) \tag{20}$$

An appropriate $k > 0$ has to be chosen such that the relation (20) is not zero. If $U_1(x) = \cos(x)$ and $U_1(x) = \sin(x)$ then the above generalized mixed spline coincides with the trigonometric mixed spline and as $k \to 0$ the mixed spline tends to the classical cubic spline provided,

$$\lim_{k \to 0} \frac{\overline{U}_2(kx)}{\overline{U}_1(kx)}$$

is non-zero.
For the sake of completeness, the idea of further generalizing the concept of mixed splines has been included. This is expected to give accurate results for a more general class of functions, than what has been derived and discussed in this paper. In fact, a deeper study is yet to be done on the choice of the functions $U_1(kx)$ and $U_2(kx)$ for any given function $f(x)$. It is intended to take up such studies in future, along with the procedure for choosing k-values based on a similar kind of minimum-norm property.

3. Minimum norm type property

This is a simple extension of the already existing minimum norm property of the classical spline approximation.

Consider the mixed spline function $\tilde{S}_\Delta(Y,x)$ which interpolates $f(x)$ at the nodes $x_0, x_1, \cdots, x_n$. Then the relation

$$\left\| f - \tilde{S}_\Delta \right\|^2 = \| f \|^2 - \left\| \tilde{S}_\Delta \right\|^2 \tag{21}$$

holds good under the following conditions:

(a) $\tilde{S}_\Delta''(Y,a) + k^2 \tilde{S}_\Delta(Y,a) = \tilde{S}_\Delta''(Y,b) + k^2 \tilde{S}_\Delta(Y,b) = 0$ - natural splines

(b) $\tilde{S}'_\Delta(Y,a) = \tilde{S}'_\Delta(Y,b)$ - Hermite splines

(c) $\tilde{S}_\Delta''(Y,a) + k^2 \tilde{S}_\Delta(Y,a) = \tilde{S}_\Delta''(Y,b) + k^2 \tilde{S}_\Delta(Y,b) = 0$ with $\tilde{S}'_\Delta(Y,a) = \tilde{S}'_\Delta(Y,b)$ and also $f'(a) = f'(b)$

(d) A k such that $M_j = -M_{j+1}$.

It is observed that condition (d), in general, makes $\tilde{S}_\Delta''(Y,x)$ discontinuous at the nodes.

We define the norm as $\| . \|^2$

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$$\|f\|^2 = \int_a^b \left|k^2 f(x) + f''(x)\right|^2 dx \tag{22}$$

We start with

$$\left\| f - \tilde{S}_\Delta \right\|^2 = \int_a^b \left|k^2 f(x) + f''(x) - k^2 \tilde{S}_\Delta(Y,x) - \tilde{S}''_\Delta(Y,x)\right|^2 dx \tag{23}$$

and consider

$$I_j = \int_{x_j}^{x_{j+1}} \left(k^2 f(x) + f''(x) - k^2 \tilde{S}_\Delta(Y,x) - \tilde{S}''_\Delta(Y,x)\right) dx \tag{24}$$

The relation (24) can be reduced to the form

$$\begin{aligned} I_j = {} & \int_{x_j}^{x_{j+1}} \left(k^2 f(x) + f''(x)\right)^2 dx - \int_{x_j}^{x_{j+1}} \left(k^2 \tilde{S}_\Delta(Y,x) + \tilde{S}''_\Delta(Y,x)\right)^2 dx \\ & - 2 \int_{x_j}^{x_{j+1}} \left(k^2 f(x) + f''(x) - k^2 \tilde{S}_\Delta(Y,x) - \tilde{S}''_\Delta(Y,x)\right) \times \left(k^2 \tilde{S}_\Delta(Y,x) + \tilde{S}''_\Delta(Y,x)\right) dx \end{aligned} \tag{25}$$

The last integral in the relation (25) can be re-written as

$$\begin{aligned} & k^2 \int_{x_j}^{x_{j+1}} \left(f(x) - \tilde{S}_\Delta(Y,x)\right) \times \left(k^2 \tilde{S}_\Delta(Y,x) + \tilde{S}''_\Delta(Y,x)\right) dx \\ & + \int_{x_j}^{x_{j+1}} \left(f''(x) - \tilde{S}''_\Delta(Y,x)\right) \times \left(k^2 \tilde{S}_\Delta(Y,x) + \tilde{S}''_\Delta(Y,x)\right) dx \end{aligned} \tag{26}$$

Integrating the second integral appearing in the relation (26) by parts and summing up over $j = 0,1,\cdots,n-1$ and using any of the conditions (a) to (c), as listed above, one can show that the value of the integral reduces to zero. Thus, we arrive at the relation

$$\begin{aligned} I_j = {} & \int_{x_j}^{x_{j+1}} \left(k^2 f(x) + f''(x)\right)^2 dx - \int_{x_j}^{x_{j+1}} \left(k^2 \tilde{S}_\Delta(Y,x) + \tilde{S}''_\Delta(Y,x)\right)^2 dx \\ & - 2k^2 \int_{x_j}^{x_{j+1}} \left(f(x) - \tilde{S}_\Delta(Y,x)\right) \times \left(k^2 \tilde{S}_\Delta(Y,x) + \tilde{S}''_\Delta(Y,x)\right) dx \end{aligned} \tag{27}$$

Summing up over $j = 0,1,\cdots,n-1$ we finally arrive at

$$\left\| f - \tilde{S}_\Delta \right\|^2 = \|f\|^2 - \left\|\tilde{S}_\Delta\right\|^2 - 2k^2 \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \left(f(x) - \tilde{S}_\Delta(Y,x)\right) \times \left(k^2 \tilde{S} \right. \tag{28}$$

In any subinterval $\Delta = [x_j, x_{j+1}]$, if we assume that the linear function $k^2 \tilde{S}_\Delta(Y,x) + \tilde{S}''_\Delta(Y,x)$ do not change sign, then by the weighted mean value theorem of integral calculus, we get

$$\left\| f - \tilde{S}_\Delta \right\|^2 = \|f\|^2 - \left\|\tilde{S}_\Delta\right\|^2 - 2k^2 \sum_{j=0}^{n-1} \left(f(\eta_j) - \tilde{S}_\Delta(Y,\eta_j)\right) \int_{x_j}^{x_{j+1}} \left(k^2 \tilde{S}_\Delta(Y,x) + \tilde{S}''_\Delta(Y,x)\right) dx \tag{29}$$

for some $\eta_j \in [x_j, x_{j+1}]$.

It can be verified on integrating by parts, that

$$\int_{x_j}^{x_{j+1}} \left(k^2 \tilde{S}_\Delta(Y,x) + \tilde{S}''_\Delta(Y,x)\right) dx = \frac{M_{j+1} + M_j}{2k^2} h_{j+1} \tag{30}$$

Now suppose we choose a k such that

$$M_{j+1} = -M_j \text{ for } j = 0,1,\cdots,n-1, \tag{31}$$

we arrive at the relation (21). It is then noted that in each subinterval Δ a different k_j is used and this makes $\tilde{S}''_\Delta(Y,x)$ discontinuous at the nodes.

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4. Uniqueness property

Using the conditions $k^2\tilde{S}_\Delta(Y,a)+\tilde{S}''_\Delta(Y,a)=k^2\tilde{S}_\Delta(Y,b)+\tilde{S}''_\Delta(Y,b)=0$ we can determine the moments $M'_j s$ completely and hence we can approximate $f(x)$ by $\tilde{S}_\Delta(Y,x)$ uniquely, if the determinant of the tridiagonal matrix appearing in the relation (14) is non-singular.

We suppose that the relation (21) holds good. If $\tilde{S}_\Delta(Y,x)$ and $\tilde{\tilde{S}}_\Delta(Y,x)$ are two mixed splines approximating $f(x)$, along with the conditions that

$$\left.\begin{array}{l}\tilde{S}_\Delta(Y,x_j)=f(x_j)\\ \tilde{\tilde{S}}_\Delta(Y,x_j)=f(x_j)\\ k^2\tilde{S}_\Delta(Y,a)+\tilde{S}''_\Delta(Y,a)=0\\ k^2\tilde{S}_\Delta(Y,b)+\tilde{S}''_\Delta(Y,b)=0\\ k^2\tilde{\tilde{S}}_\Delta(Y,a)+\tilde{\tilde{S}}''_\Delta(Y,a)=0\\ k^2\tilde{\tilde{S}}_\Delta(Y,b)+\tilde{\tilde{S}}''_\Delta(Y,b)=0\end{array}\right\} \tag{33}$$

then we arrive at

$$\left\| f-\tilde{S}_\Delta \right\|^2=\|f\|^2-\left\|\tilde{S}_\Delta\right\|^2 \tag{34}$$

and also

$$\left\| f-\tilde{\tilde{S}}_\Delta \right\|^2=\|f\|^2-\left\|\tilde{\tilde{S}}_\Delta\right\|^2 \tag{35}$$

Replacing $f(x)$ by $\tilde{\tilde{S}}_\Delta(Y,x)$ in the relation (34), we obtain

$$\left\| \tilde{\tilde{S}}_\Delta-\tilde{S}_\Delta \right\|^2=\left\|\tilde{\tilde{S}}_\Delta\right\|^2-\left\|\tilde{S}_\Delta\right\|^2. \tag{36}$$

Interchanging $\tilde{\tilde{S}}_\Delta(Y,x)$ and $\tilde{S}_\Delta(Y,x)$ in the relation (36), we arrive at

$$\left\| \tilde{\tilde{S}}_\Delta-\tilde{S}_\Delta \right\|^2=\left\|\tilde{\tilde{S}}_\Delta\right\|^2-\left\|\tilde{S}_\Delta\right\|^2. \tag{37}$$

Adding the two relations (36) and (37), we get

$$\left\| \tilde{\tilde{S}}_\Delta-\tilde{S}_\Delta \right\|^2=0 \tag{38}$$

If a constant k exists such that condition (32) is satisfied as well as $\tilde{\tilde{S}}''_\Delta(Y,x)$ and $\tilde{S}''_\Delta(Y,x)$ are continuous, by the definition of $\|.\|^2$, we obtain that

$$\left(\tilde{\tilde{S}}''_\Delta(Y,x)-\tilde{S}''_\Delta(Y,x)\right)+k^2\left(\tilde{\tilde{S}}_\Delta(Y,x)-\tilde{S}_\Delta(Y,x)\right)=0 \tag{39}$$

Using the conditions (33), we can then conclude that

$$\tilde{S}_\Delta(Y,x)\equiv\tilde{\tilde{S}}_\Delta(Y,x) \tag{40}$$

This proves the uniqueness.

The generality involved in the tridiagonal system of equations (14), makes the direct verification of the fact that as the parameter $k\to 0$, the mixed spline function tends to the classical cubic spline formula: $\tilde{S}_\Delta(Y,x)\to S_\Delta(Y,x)$. But the result is verified for $n=2$ and $n=3$.

5. Development of quadrature rules

By direct integration of $\tilde{S}_\Delta(Y,x)$ on the respective subintervals we can derive various quadrature formulae. Thus we derive the following expressions after integration of $\tilde{S}_\Delta(Y,x)$ over the interval $[x_j,x_{j+1}]$:

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$$I_j = \int_{x_{j-1}}^{x_j} \tilde{S}_\Delta(Y,x)dx = \left(y_j - \frac{M_j}{k^2} + y_{j-1} - \frac{M_{j-1}}{k^2}\right)\left(\frac{1-\cos(\theta)}{k\sin(\theta)}\right) + \frac{h}{2k^2}\left(M_j + M_{j-1}\right) \tag{41}$$

where $\theta = kh$. Summing over $j = 1,2,\cdots,n$ we arrive at

$$\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} \tilde{S}_\Delta(Y,x)dx = \sum_{j=1}^{n} I_j \tag{42}$$

which is an approximation for $I = \int_a^b f(x)dx.$ It is also verified that as $k \to 0,$ the new quadrature formulae tend to the corresponding classical results.

6. Numerical study

We have considered an equidistant set of points $0, h, 2h, \cdots, nh,$ where $h = \frac{b-a}{n}$. We have worked with the values of $n = 12, 14, 16, 18, 20, 22, 24, 26, 28, 30$. Various test functions are chosen for numerical study. A few functions are tested for the validity of the approximation and a few others for testing the accuracy of the quadrature formulae. In both cases, it can be inferred that the mixed spline approximation or its quadrature rule, is better. We have chosen k values arbitrarily.

Table 1, Table 2 and Table 3 compare the absolute errors in the quadrature formulae based on the classical and mixed spline. Figures 1, 2 and 3 compare the accuracy in approximation of test functions using classical cubic spline and mixed cubic spline. The minimum norm property is verified for two simple functions, satisfying the requirements as discussed in section 4, and the changes are remarkable. Figures 5 and 7 verify the minimum norm property. This can be compared with Figures 4 and 6, wherein k is chosen arbitrarily.

For $n = 2, h = \frac{1}{2}, \theta = kh$ and $x \in [0,h]$ the approximation is given to be

$$\tilde{S}_\Delta(Y,x) = \left(f(0) - \frac{M_0}{k^2}\right)\frac{\sin(\theta - kx)}{\sin(\theta)} + \left(f(h) - \frac{M_1}{k^2}\right)\frac{\sin(kx)}{\sin(\theta)} + \frac{1}{k^2}\left(\frac{x}{h}M_1 + \frac{h-x}{h}M_0\right) \tag{43}$$

and for $x \in [h,2h]$ we obtain

$$\tilde{S}_\Delta(Y,x) = \left(f(h) - \frac{M_1}{k^2}\right)\frac{\sin(2\theta - kx)}{\sin(\theta)} + \left(f(2h) - \frac{M_2}{k^2}\right)\frac{\sin(kx - \theta)}{\sin(\theta)} + \frac{1}{k^2}\left(\frac{x-h}{h}M_2 + \frac{2h-x}{h}M_1\right) \tag{44}$$

Similarly, for $n = 3, h = \frac{1}{3},$ and $x \in [0,h]$ we obtain

$$\tilde{S}_\Delta(Y,x) = \left(f(0) - \frac{M_0}{k^2}\right)\frac{\sin(\theta - kx)}{\sin(\theta)} + \left(f(h) - \frac{M_1}{k^2}\right)\frac{\sin(kx)}{\sin(\theta)} + \frac{1}{k^2}\left(\frac{x}{h}M_1 + \frac{h-x}{h}M_0\right) \tag{45}$$

for $x \in [h,2h]$ we can derive

$$\tilde{S}_\Delta(Y,x) = \left(f(h) - \frac{M_1}{k^2}\right)\frac{\sin(2\theta - kx)}{\sin(\theta)} + \left(f(2h) - \frac{M_2}{k^2}\right)\frac{\sin(kx - \theta)}{\sin(\theta)} + \frac{1}{k^2}\left(\frac{x-h}{h}M_2 + \frac{2h-x}{h}M_1\right) \tag{46}$$

and for $x \in [2h,3h]$ we get

$$\tilde{S}_\Delta(Y,x) = \left(f(2h) - \frac{M_2}{k^2}\right)\frac{\sin(3\theta - kx)}{\sin(\theta)} + \left(f(3h) - \frac{M_3}{k^2}\right)\frac{\sin(kx - 2\theta)}{\sin(\theta)} + \frac{1}{k^2}\left(\frac{x-2h}{h}M_3 + \frac{3h-x}{h}M_2\right) \tag{47}$$

It is verified that the expressions (43)-(47) tend to the corresponding classical cubic spline interpolation functions, as the parameter $k \to 0.$

Table 1 $\int_0^{\pi} xe^x \sin(5x)dx = 13.6234, k = 5$

n	Classical Spline	Mixed Spline
12	3.66529	0.205979
14	2.76146	0.089989
16	2.14911	0.0367852
18	1.7177	0.0105582
20	1.40328	0.00290153

Table 2 $\int_0^{\pi} \cos(4x)\sin(5x)dx = 1.111111, k = 3$

n	Classical Spline	Mixed Spline
12	0.104116	0.00930373
14	0.0812809	0.00234235
16	0.0636427	0.000516667
18	0.0507395	0.000008359
20	0.0412523	0.000125106

Table 3 $\int_0^{\pi} \frac{\sin(3x)}{1+25x^2} dx = 0.124439, k = 4.8$

n	Classical Spline	Mixed Spline
20	0.0177886	0.00114583
22	0.0150026	0.000688031
24	0.0128142	0.000395943
26	0.0110667	0.000204782
28	0.00965006	0.0000781875
30	0.00848596	0.0000056522

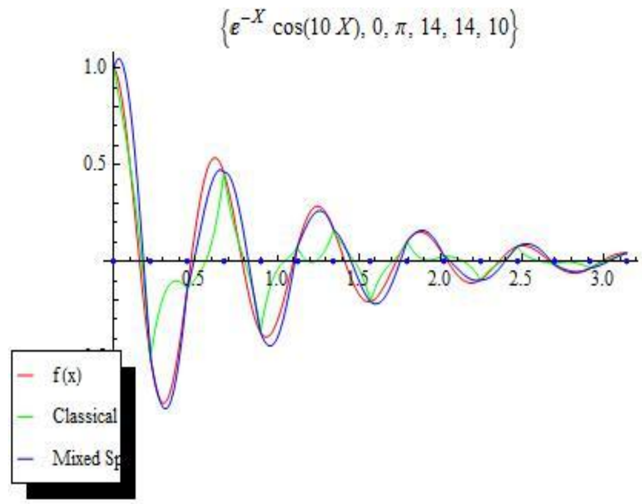

Figure 1: Comparison between classical cubic spline and mixed spline. Interval $[0, \pi]$**, Number of subintervals** $= 14, k = 10$

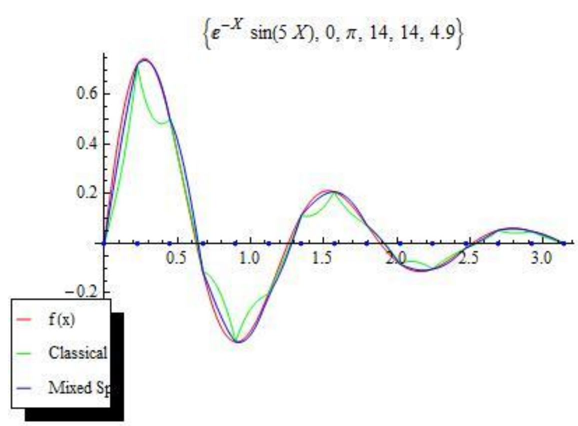

Figure 2: Comparison between classical cubic spline and mixed spline. Interval $[0, \pi]$**, Number of subintervals** $= 14, k = 4.9$

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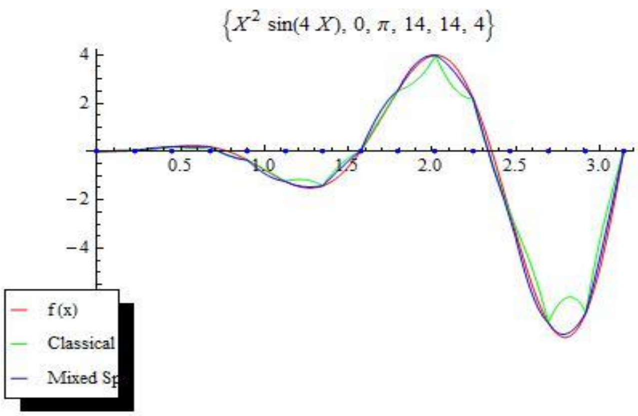
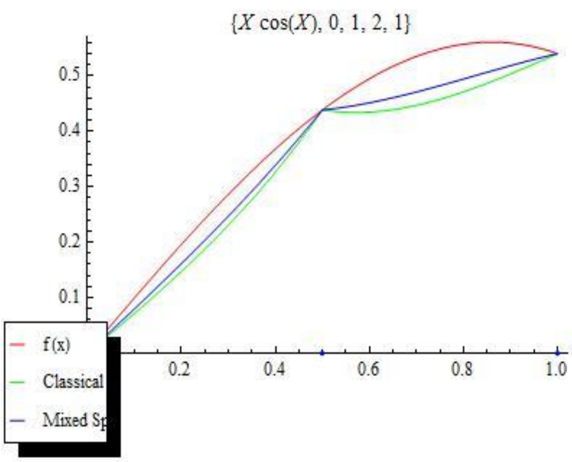


Figure 3: Comparison between classical cubic spline and mixed spline. Interval $[0, \pi]$, Number of subintervals $= 14, k = 4$



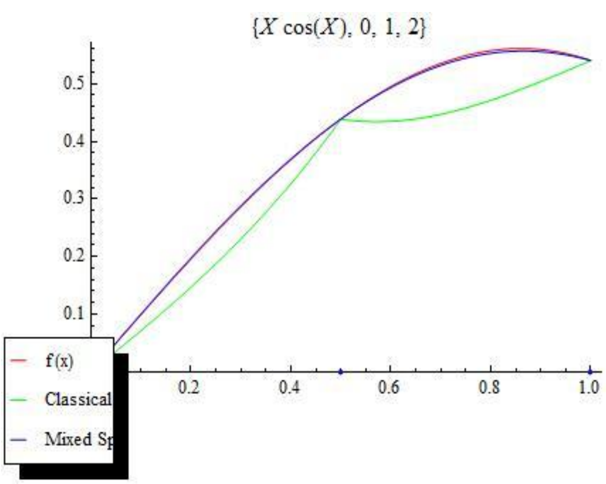
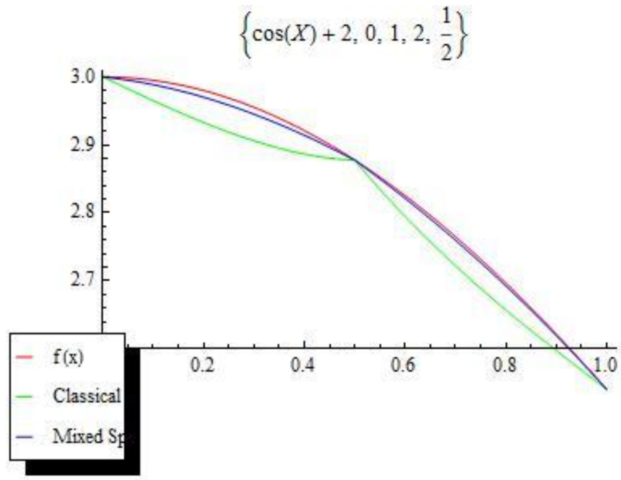


Figure 4: $x \in [0,1], n = 2, k = 1$

Figure 5: $x \in [0,1], n = 2, k$ satisfies minimum norm



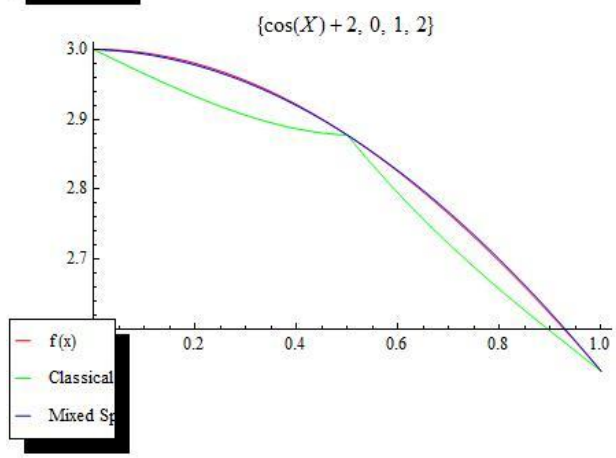


Figure 6: $x \in [0,1], n = 2, k = \frac{1}{2}$

Figure 7: $x \in [0,1], n = 2, k$ satisfies minimum norm

7 Conclusions

The classical cubic splines have been generalized based on the idea of mixed interpolation, which can be further extended to higher order splines. The mixed splines have been derived by introducing trigonometric functions along with a linear function. A minimum norm type property for the mixed splines has been established at the cost of a discontinuity of the second derivative of the mixed spline at the nodal values. The uniqueness of the mixed spline approximation has been proved at the same cost. Also, quadrature rules have been derived by integrating the mixed spline approximation. A few examples have been considered for the purpose of comparison. It has been verified that the approximation formula, as well as the quadrature rule based on mixed spline, tend to the corresponding classical formulae in the limit as $k \to 0$.

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