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Study on Prime n-ideals of a Lattice

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Abstract— For a fixed element n in a lattice L , any convex sub-lattice containing *n* is called an n –ideal.We generalize several results of prime ideals of a lattice in terms of *n*-ideals. We introduce the notion of relative prime n-ideals in a lattice and include some interesting results on this.

Keywords—n-ideals, Prime n- ideals, Convex sub-lattice, distributive lattice.

I. Introduction

The idea of n-ideals in a lattice was first introduced by Cornish and Noor in several Papers vii, viii, ix, x,xi. For a fixed element n of a lattice L, a convex sub lattice containing n is called an n-ideal. If L has a '0' then replacing n by 0, an n-ideals becomes an ideal. Moreover, if L has a '1', an n-ideals becomes a filter by replacing n by '1'. Thus the idea of n-ideals is a kind of generalization of both ideals and filters of a lattice. So any results involving n-ideals will give a generalization of the results on ideals and filters with 0 and 1 respectively in a lattice. The set of all n-ideals of a lattice L is denoted by $I_n(L)$, which is an algebraic lattice under set-inclusion. Moreover, {L} and L are respectively the smallest and largest elements of $I_n(L)$, while the set–theoretic intersection is the infimum.

II. Material and Methodology

For any two n-ideals I and J of L, it is easy to check that
$I \wedge J = I \cap J = \{x: x= m(i, n, j) \text{ for some } i \in I, j \in J\}$
Where $m(x,y,z)=(x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$ and
$I\vee J = \{x: i_1\wedge j_1 \le x \le i_2\vee j_2, \text{ for some } i_1,i_2 \in I \text{ and } j_1,j_2\in J\}$.
The n-ideal generated by $a_1, a_2, \ldots\ldots. a_m$ is denoted by
$\langle a_1, a_2, \ldots\ldots.a_m \rangle_n$.
Clearly $\langle a_1, a_2, \ldots\ldots.a_m \rangle_n = \langle a_1\rangle_n \vee \langle a_2\rangle_n \vee \ldots\ldots.\vee\langle a_m\rangle_n$. The n-ideal generated by a finite number of elements is called a finitely generated n-ideal. The set of all finitely generated n-ideal is denoted by $F_n(L)$. Also the n-ideal generated by a single element is called a principal n-ideal. The set of all principal n-ideal of L is denoted by $P_n(L)$. We have $\langle a\rangle_n = \{x\in L: a\wedge n \le x \le a\vee n\}$.
The median operation $m(x, y, z)=(x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$ is very well known in lattice theory.
An n-ideal P of a lattice L is called prime if $m(x, n, y) \in P; x,y\in L$ implies either $x\in P$ or $y\in P$.
An element s of a lattice L is called standard if for all $x, y\in L$, $x\wedge(y\vee s) = (x\wedge y)\vee(x\wedge s)$. An element $n\in L$ is called neutral if it is standard and for all $x, y\in L$, $n\wedge(x\vee y) = (n\wedge x)\vee(n\wedge y)$. Of course 0 and 1 of a lattice are always neutral. An element $n\in L$ is called central if it is neutral and complimented in each interval containing n.
A lattice L with 0 is called sectionally complimented if [0,x] is complimented for all $x\in L$.
A sub-lattice H of a lattice L is called a **convex sub-lattice** if for any $a,b\in H$, $a< c < b$ implies $c\in H$.
Example: in the fig below(fig-1) {0,a,c},{0,b,c} are convex lattices.

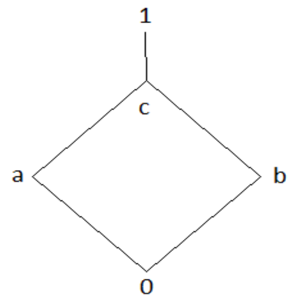

fig-1: Convex Lattice

III. Results and Tables

Theorem 3.1: If P is a prime n-ideal of a lattice, then for any $x\in L$, at least one of $x \wedge n$ and $x \vee n$ is a member of P.

Proof: Observe that $m(x \wedge n, n, x \vee n) = n \in P$.
Thus either $x\wedge n \in P$ or $x\vee n \in P$.

Theorem 3.2: If P is a prime n-ideal of a lattice, then P contains either (n] or [n), but not both.
Proof: Suppose P is prime and $P \not\supseteq (n]$. Then there exist $r < n$ such that $r \notin P$. Now let $s\in[n)$. Then $m(r, n, s) = (r\wedge n)\vee(n\wedge s)\vee(s\wedge r) = r\vee n\vee r = n\in P$ implies that $s\in P$.
That is, $P\supseteq[n)$.
Similarly, if $P \not\supseteq [n)$, then we can show $P \supseteq (n]$.
Finally suppose that P contains both (n] and [n). Let $t\in L$.
Then $t\wedge n \in P$ and $t\vee n \in P$.
Then by convexity of n-ideals $t \in P$. This implies P=L, which is a contradiction to the primeness of P.

Corollary 3.3: If P is a prime n-ideal of a lattice L, then there exists at least one $x\in L$ such that both $x \wedge n$ and $x \vee n$ do not belong to P.

Theorem 3.4: Let n be a neutral element of a lattice L. Then an n-ideal P is Prime if and only if it is a prime ideal or a prime dual ideal (filter).

Proof: Suppose the n-ideal P is prime. Then by theorem 2.2 either $P \supseteq (n]$ or $P \supseteq [n)$. Suppose $P \supseteq (n]$. Let $x \in P$ and $t \leq x$, $t \in L$. Then $t \wedge n \in (n] \subseteq P$.
Thus, by convexity of P, $t \wedge n \leq t \leq x$ implies that $t \in P$. This implies that P is an ideal. Also let $a \wedge b \in P$, $a,b \in L$. Then $(a \wedge b) \vee n \in P$ and $m(a, n, b)=(a \wedge n) \vee (n \wedge b) \vee (b \wedge a) \leq (a \wedge b) \vee a$ implies that $m(a, n, b) \in P$. Thus either $a \in P$ or $b \in P$, and so P is a prime ideal. On the other hand if $P \supseteq [n)$, we can similarly prove that P is a prime dual ideal. We omit the proof of the converse in trivial.

Lemma 3.5: In a distributive lattice L, a prime ideal containing n is also a prime n-ideal.
Dually we can easily prove the following result.

Lemma 3.6: In a distributive lattice L, a prime dual ideal (filter) containing n is also a prime n-ideal.

Theorem 3.7: Let L be a distributive lattice. Let I be an ideal. Let D be a dual ideal of L, and let $I \cap D = \phi$, then there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \phi$.

Proof: Following result is an improvement of above theorem which is due to [vii, Theorem 3.3].

Theorem 3.8: Let L be a distributive lattice, let I be an ideal, let D be a convex sub lattice of L and let $I \cap D = \phi$, then there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \phi$.

Now we give a separation property for distributive lattices in terms of prime n-ideals which is of course an extension of Stone's representation theorem. It should be mentioned that this result has also been obtained by Latif and Noor in [viii]. Here we include a separate proof as it much more simpler than that of [viii].

Theorem 3.9: In a distributive lattice L, suppose I is an n-ideal and D is a convex sub lattice of L with $I \cap D = \phi$. Then there exists a prime n-ideal P of L such that $P \supseteq I$ and $P \cap D = \phi$.

Proof: Since $I \cap D = \phi$, so either $(I] \cap D = \phi$ or $[I) \cap D = \phi$. If $(I] \cap D = \phi$, then by theorem 2.8, there exists a prime ideal $P \supseteq I$ such that $P \cap D = \phi$. Similarly if $[I) \cap D = \phi$, then there exists a prime filter $Q \supseteq [I)$ such that $Q \cap D = \phi$.
But by lemma 3.5 and lemma 3.6, both P and Q are prime n-ideals.

Corollary 3.10: Every n-ideal I of a distributive lattice L is the intersection of all prime n-ideals containing it.
Proof: Let $I_1 = \cap\{P: P \supseteq I, P$ is a prime n-ideal of L$\}$. If $I = I_1$, then there is an element $a \in I_1 - I$. Then by above corollary, there is a prime n-ideal P with $P \supseteq I$, $a \notin P$. But, $a \notin P \supseteq I$ gives a contradiction.

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