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Orthogonal Hybrid Functions (HF) for Solving Second Order Differential Equations using One-Shot Integration Operational Matrices

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Abstract— The present work employs a new set of orthogonal hybrid functions (HF) generated from the synthesis of orthogonal sample-and-hold functions (SHF) and triangular functions (TF).

The one-shot operational matrices for second order integration are derived. These matrices are employed for more accurate second order integration. Finally, these matrices are employed for solving second order non-homogeneous differential equations followed by a numerical example. The results are compared with the exact solution as well as the results obtained via 4th order Runge-Kutta method.

Keywords: Hybrid functions, Sample-and-hold functions, Triangular functions, Function approximation, Operational matrices, Differential equation

I. INTRODUCTION

For more than three decades different piecewise constant basis functions (PCBF) have been employed to solve problems in different fields of engineering including control theory. It was in 1910, when Haar functions [1] appeared as the first set of the PCBF family. As far as shapes were concerned, this function set was entirely different from the 'orthodox' sine-cosine functions and was the genesis of a new class of orthogonal functions. Piecewise constant nature of this 'new' class of functions attracted many researchers to explore its appropriate application areas. Of this class, the block pulse function (BPF) [2, 3] set and its variants [4] proved to be the most efficient because of its simplicity and versatility in analysis [5] as well as synthesis [4, 6] of control systems.

In 1998, an orthogonal set of sample-and-hold functions [7] were introduced by Deb et al and the same was applied to solve problems related to discrete time systems with zero order hold. The set of sample-and-hold functions approximate any square integrable function of Lebesgue measure in a piecewise constant manner and was proved to be more convenient for solving problems associated to sample-and-hold systems.

In 2003, orthogonal triangular functions [8] were introduced by Deb et al and the same were applied to control system related problems including analysis and system identification. The set of triangular functions approximate any square integrable function in a piecewise linear manner.

In this paper, a new set of orthogonal hybrid functions (HF), which is a combination of sample-and-hold function and triangular function, is presented. This new function set is advantageous for

(i) Function approximation,
(ii) Computation of the operational matrix for integration in hybrid function (HF) domain,
(iii) Integration of time functions using the operational matrix for integration,
(iv) Computation of one-shot operational matrices for integration of second order,
(v) Second order integration of time functions using one-shot operational matrices,
(vi) Solution of linear second order differential equations using one-shot operational matrices.

II. HYBRID FUNCTION (HF) : A COMBINATION OF SHF AND TF

We can use a set of sample-and-hold functions and the RHTF set of triangular functions to form a hybrid function set, which we name a 'Hybrid function set'. To define a hybrid function (HF) set, we express the i-th member $H_i(t)$ of the m-set hybrid function $\mathbf{H_{(m)}(t)}$ in $0 \le t < T$ as

$$H_i(t)(a_i, b_i) = a_i S_i(t) + b_i T2_i(t) \tag{1}$$

where, i = 0, 1, 2, …, (m-1), a_i and b_i are scaling constants. For convenience, in the following, we write **T** instead of **T2**.

The hybrid function set always comes up with a piecewise linear solution. Fig. 1 illustrates how a function f(t) is represented via hybrid functions.

The function f(t) is sampled at three equidistant points (sampling interval h) at A, C and E with sample values c_0 , c_1 and c_2 . Now, f(t) can be expressed in a piecewise linear form by the two straight lines AC and CE, which are the sides of two adjacent trapeziums. Then

$$f(t) \approx H_0(t) + H_1(t)$$
$$= \{c_0 S_0(t) + (c_1 - c_0) T_0(t)\} + \{c_1 S_1(t) + (c_2 - c_1) T_1(t)\}$$
$$= \{c_0 S_0(t) + c_1 S_1(t)\} + \{(c_1 - c_0) T_0(t) + (c_2 - c_1) T_1(t)\}$$
$$= \mathbf{C}^T \mathbf{S}_{(2)}(t) + \mathbf{D}^T \mathbf{T}_{(2)}(t)$$

where, $[c_0 \quad c_1] \triangleq \mathbf{C}^T$ and

$[(c_1 - c_0) \quad (c_2 - c_1)] \triangleq \mathbf{D}^T$

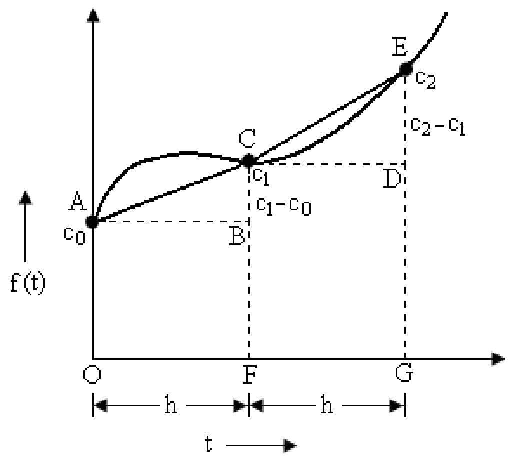

Fig. 1 : A function f(t) represented via hybrid functions

III. INTEGRATION OF FUNCTIONS USING HYBRID FUNCTION DOMAIN OPERATIONAL MATRICES

Let f(t) be a square integrable function which can be expanded in hybrid function domain as

$$f(t) \approx [c_0 \quad c_1 \quad c_2 \quad \cdots \quad c_{m-1}] \mathbf{S}_{(m)}(t)$$
$$+ [(c_1 - c_0) \quad (c_2 - c_1) \quad (c_3 - c_2) \quad \cdots \quad (c_m - c_{m-1})] \mathbf{T}_{(m)}(t)$$
$$= \mathbf{C}_\mathbf{S}^T \mathbf{S}_{(m)}(t) + \mathbf{C}_\mathbf{T}^T \mathbf{T}_{(m)}(t) \qquad (2)$$

where, T denotes transpose.

Integrating equation (2) with respect to t, we get

$$\int f(t)\,dt \approx \int \mathbf{C}_\mathbf{S}^T \mathbf{S}_{(m)} dt + \int \mathbf{C}_\mathbf{T}^T \mathbf{T}_{(m)} dt$$
$$= \mathbf{C}_\mathbf{S}^T \int \mathbf{S}_{(m)} dt + \mathbf{C}_\mathbf{T}^T \int \mathbf{T}_{(m)} dt$$
$$= [\mathbf{C}_\mathbf{S}^T + \frac{1}{2}\mathbf{C}_\mathbf{T}^T] \int \mathbf{S}_{(m)} dt$$
$$\because \int \mathbf{T}_{(m)} dt = \frac{1}{2} \int \mathbf{S}_{(m)} dt$$

$$= \mathbf{C}_\mathbf{S}^T \begin{bmatrix} \mathbf{P1ss}_{(m)} & \vdots & \mathbf{P1st}_{(m)} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{(m)} \\ \cdots\cdots \\ \mathbf{T}_{(m)} \end{bmatrix} + \mathbf{C}_\mathbf{T}^T \begin{bmatrix} \mathbf{P1ts}_{(m)} & \vdots & \mathbf{P1tt}_{(m)} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{(m)} \\ \cdots\cdots \\ \mathbf{T}_{(m)} \end{bmatrix} \qquad (3)$$

IV. REPEATED INTEGRATION USING 1st ORDER INTEGRATION MATRICES ONLY

$$\int \mathbf{S}_{(m)} dt = \mathbf{P1ss}_{(m)} \mathbf{S}_{(m)} + \mathbf{P1st}_{(m)} \mathbf{T}_{(m)} \quad \text{and}$$
$$\int \mathbf{T}_{(m)} dt = \mathbf{P1ts}_{(m)} \mathbf{S}_{(m)} + \mathbf{P1tt}_{(m)} \mathbf{T}_{(m)}$$

So, we can write

$$\iint \mathbf{S}_{(m)} dt = \mathbf{P1ss}_{(m)} \int \mathbf{S}_{(m)} dt + \mathbf{P1st}_{(m)} \int \mathbf{T}_{(m)} dt$$
$$= [\mathbf{P1ss}_{(m)} + \frac{1}{2}\mathbf{P1st}_{(m)}] \int \mathbf{S}_{(m)} dt = \mathbf{P}_{(m)} \int \mathbf{S}_{(m)} dt \qquad (4)$$
$$\because \int \mathbf{T}_{(m)} dt = \frac{1}{2} \int \mathbf{S}_{(m)} dt \quad \text{and}$$
$$\mathbf{P1ss}_{(m)} + \frac{\mathbf{P1st}_{(m)}}{2} = \mathbf{P}_{(m)}$$

where, $\mathbf{P}_{(m)}$ is the block pulse operational matrix for integration [2] of order m.

Thus, for n times repeated integration of the $\mathbf{S}_{(m)}$ vector, we have

$$\underbrace{\int\int\int\cdots\int}_{n} \mathbf{S_{(m)}}dt = \mathbf{P_{(m)}}^{(n-1)}\int \mathbf{S_{(m)}}dt$$

where, n = 2, 3, 4, …, n(say) (5)

Hence, for the vector $\mathbf{T_{(m)}}$ it can be shown that

$$\int\int \mathbf{T_{(m)}}dt = \mathbf{P1ts_{(m)}}\int \mathbf{S_{(m)}}\,dt + \mathbf{P1tt_{(m)}}\int \mathbf{T_{(m)}}\,dt$$

$$= \frac{1}{2}\mathbf{P1ss_{(m)}}\int \mathbf{S_{(m)}}\,dt + \frac{1}{2}\mathbf{P1st_{(m)}}\int \mathbf{T_{(m)}}\,dt$$

$$= \mathbf{P1ss_{(m)}}\int \mathbf{T_{(m)}}\,dt + \frac{1}{2}\mathbf{P1st_{(m)}}\int \mathbf{T_{(m)}}\,dt$$

$$= \mathbf{P_{(m)}}\int \mathbf{T_{(m)}}\,dt$$

Thus, for n times repeated integration

$$\underbrace{\int\int\int\cdots\int}_{n} \mathbf{T_{(m)}}dt = \mathbf{P_{(m)}}^{(n-1)}\int \mathbf{T_{(m)}}dt = \frac{\mathbf{P_{(m)}^{(n-1)}}}{2}\int \mathbf{S_{(m)}}dt = \frac{1}{2}\underbrace{\int\int\int\cdots\int}_{n} \mathbf{S_{(m)}}dt$$

for n = 1, 2, 3,…n (say) (6)

V. ONE-SHOT INTEGRATION OPERATIONAL MATRICES FOR REPEATED INTEGRATION

The result of integration is somewhat approximate via the operation of first order integration using operational matrices $\mathbf{P1ss}$, $\mathbf{P1st}$, $\mathbf{P1ts}$, $\mathbf{P1tt}$. If we carry on repeated integration using these matrices, error will surely accumulate and higher order integrations in HF domain may become so corrupted that it may lead to a fiasco.

For this reason, we present in the following more accurate one-shot operational matrices of higher orders suitable for computation of repeated integration of functions with improved accuracy.

The basic principle of determination of one-shot operational matrices for integration is elaborated by the following steps:

(i) Integrate the sample-and-hold basis function set repeatedly 2 times.
Find out the samples of the 2 times integrated curves.

(ii) From these samples, form corresponding sample-and-hold function coefficient row matrices as well as the triangular function coefficient row matrices. That is, the 2 times integrated function is expressed in HF domain.

(iii) Integrate the triangular basis function set twice.

(iv) Find out the samples of the 2 times integrated curves.

(v) From these samples, form corresponding sample-and-hold function coefficient row matrices and the triangular function coefficient row matrices. That is, the 2 times integrated function is expressed in HF domain.

(vi) From the above steps, form one-shot operational matrices of 2nd order integration.

A. *One-shot operational matrix for sample-and-hold functions*

(i) Second order matrices

The first member S_0 of the SHF set is integrated twice and Fig. 2 shows the integrated function $\int\int S_0 dt$.

The samples of the resulting function at sampling instants 0, h, 2h, 3h and 4h are

$0, \frac{h^2}{2}, \{\frac{h^2}{2} + h(2h-h)\}, \{\frac{h^2}{2} + h(3h-h)\}$ and

$\{\frac{h^2}{2} + h(4h-h)\}$ respectively.

From these samples we develop the one-shot operational matrix for double integration for m = 4 as

$$\int\int \mathbf{S_{(4)}}dt = \left[\begin{array}{ccc} \mathbf{P2ss_{(4)}} & \vdots & \mathbf{P2st_{(4)}} \end{array}\right]\left[\begin{array}{c} \mathbf{S_{(4)}} \\ \cdots\cdots \\ \mathbf{T_{(4)}} \end{array}\right] \quad (7)$$

where, $\mathbf{P2ss_{(4)}} = \frac{h^2}{2}\begin{bmatrix}0 & 1 & 3 & 5\end{bmatrix}$

and $\mathbf{P2st}_{(4)} = \frac{h^2}{2}\begin{bmatrix} 1 & 2 & 2 & 2 \end{bmatrix}$

For m terms, the generalized one-shot operational matrices for double integration are:

$$\left.\begin{aligned} \mathbf{P2ss}_{(m)} &= \frac{h^2}{2}\big[\underbrace{0 \quad 1 \quad 3 \quad 5 \;\ldots\;\ldots\; (2m-3)}_{m\ \text{terms}}\big] \\ \text{and}\quad \mathbf{P2st}_{(m)} &= \frac{h^2}{2}\big[\underbrace{1 \quad 2 \quad 2 \quad 2 \;\ldots\;\ldots\; 2 \quad 2}_{m\ \text{terms}}\big] \end{aligned}\right\} \quad (8)$$

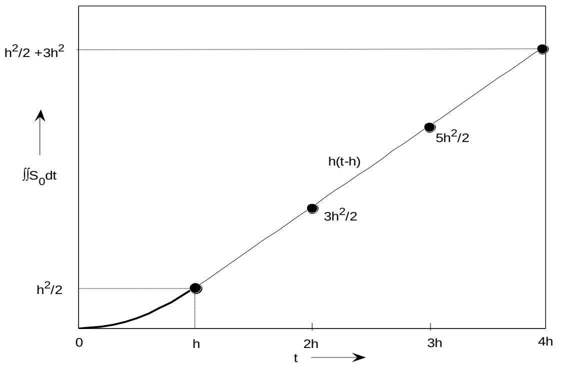

Fig. 2 : Double integration of the first member S_0 of the SHF set.

B. One-shot operational matrix for triangular functions

(i) Second order matrices

The first member T_0 of the triangular function set is integrated twice and Fig. 3 shows the integrated function $\iint T_0\, dt$

The samples of the resulting function at sampling instants 0, h, 2h, 3h and 4h are

$0,\ \frac{h^2}{6},\ \{\frac{h^2}{6} + \frac{h}{2}(2h\text{-}h)\},\ \{\frac{h^2}{6} + \frac{h}{2}(3h\text{-}h)\},$

$\{\frac{h^2}{6} + \frac{h}{2}(4h\text{-}h)\}$ respectively.

From these samples we develop the one-shot operational matrix for double integration for m = 4:

$$\iint \mathbf{T}_{(4)}\, dt = \begin{bmatrix} \mathbf{P2ts}_{(4)} & \vdots & \mathbf{P2tt}_{(4)} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{(4)} \\ \cdots\cdots \\ \mathbf{T}_{(4)} \end{bmatrix} \quad (9)$$

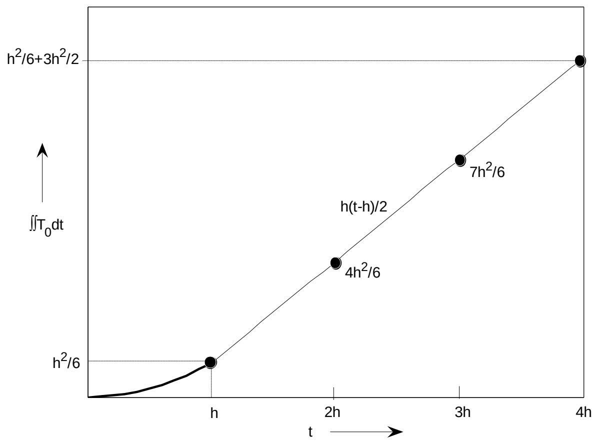

Fig. 3 : Double integration of the first member T_0 of the triangular function set.

where, $\mathbf{P2ts}_{(4)} = \frac{h^2}{6}\begin{bmatrix} 0 & 1 & 4 & 7 \end{bmatrix}$

and, $\mathbf{P2tt}_{(4)} = \frac{h^2}{6}\begin{bmatrix} 1 & 3 & 3 & 3 \end{bmatrix}$

For m terms, the generalized one-shot operational matrices for double integration are

$$\left.\begin{aligned} \mathbf{P2ts}_{(m)} &= \frac{h^2}{6}\big[\underbrace{0 \quad 1 \quad 4 \quad 7 \;\ldots\;\ldots\;\ldots\; (3m-5)}_{m\ \text{terms}}\big] \quad (10) \\ \text{and,}\quad \mathbf{P2tt}_{(m)} &= \frac{h^2}{6}\big[\underbrace{1 \quad 3 \quad 3 \quad 3 \;\ldots\;\ldots\;\ldots\; 3}_{m\ \text{terms}}\big] \end{aligned}\right\}$$

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TABLE I:
COMPARISON OF THREE SETS OF SAMPLES OF THE FUNCTION f(t) : (i) OBTAINED VIA DIRECT EXPANSION IN HF DOMAIN, (ii) OBTAINED VIA REPEATED APPLICATION OF HF DOMAIN INTEGRATION OPERATIONAL MATRICES OF 1ST ORDER ONLY AND (iii) OBTAINED VIA HF DOMAIN ONE-SHOT OPERATIONAL MATRICES OF 2ND ORDER.

t (sec)	Samples of f(t) via direct expansion in HF domain	Samples of f(t) via HF domain integration matrices of 1st order	% Error	Samples of f(t) via HF domain one-shot integration matrices of 2nd order	% Error
0	0.0000	0.0000	-	0.0000	-
$\frac{1}{8}$	0.0081	0.0083	-2.4691	0.0081	0
$\frac{2}{8}$	0.0339	0.0342	-0.8849	0.0339	0
$\frac{3}{8}$	0.0791	0.0796	-0.6321	0.0791	0
$\frac{4}{8}$	0.1458	0.1465	-0.4801	0.1458	0
$\frac{5}{8}$	0.2360	0.2368	-0.3389	0.2360	0
$\frac{6}{8}$	0.3516	0.3525	-0.2559	0.3516	0
$\frac{7}{8}$	0.4945	0.4956	-0.2224	0.4945	0
$\frac{8}{8}$	0.6667	0.668	-0.1949	0.6667	0

VI. NUMERICAL EXAMPLE 1

Let us now take up an example to compare the efficiencies of repeated use of 1st order integration matrices and second order one-shot integration matrices.
Consider the function

$$f(t) = \int t\,dt + \iint t\,dt = \frac{t^2}{2} + \frac{t^3}{6} \quad (11)$$

Let $f(t) \approx \mathbf{D}_\mathbf{S}^T \mathbf{S}_{(\mathbf{m})} + \mathbf{D}_\mathbf{T}^T \mathbf{T}_{(\mathbf{m})}$ (12)

where, $\mathbf{D}_\mathbf{S}$ and $\mathbf{D}_\mathbf{T}$ are HF domain coefficient vectors of f(t) known from actual samples of the function f(t).

Also, let $t \approx \mathbf{C}_\mathbf{S}^T \mathbf{S}_{(\mathbf{m})} + \mathbf{C}_\mathbf{T}^T \mathbf{T}_{(\mathbf{m})}$ (13)

where, $\mathbf{C}_\mathbf{S}$ and $\mathbf{C}_\mathbf{T}$ are HF domain coefficient vectors known from actual samples of the function t.
Now we perform single and double integration on the RHS of (13) via HF domain and substitute the results in (11) to obtain HF domain solution of f(t).

Considering the discussion in earlier sections, we can determine the result by performing the integration in HF domain in the following two ways :

(i) Using the 1st order HF domain integration operational matrices $\mathbf{P1ss}_{(m)}$, $\mathbf{P1st}_{(m)}$, $\mathbf{P1ts}_{(m)}$ and $\mathbf{P1tt}_{(m)}$.

(ii) Using HF domain one-shot integration operational matrices of 2nd orders.

(iii) Finally, a comparison may be made between the results obtained via above two integration methods with the exact samples of the function f(t) of equation (12).

A. By repeated use of HF domain 1st order integration matrices $\mathbf{P1ss}_{(m)}$, $\mathbf{P1st}_{(m)}$, $\mathbf{P1ts}_{(m)}$ and $\mathbf{P1tt}_{(m)}$

We know that

$$\int t\,dt = \mathbf{C}_\mathbf{S}^T \int \mathbf{S}_{(\mathbf{m})}\,dt + \mathbf{C}_\mathbf{T}^T \int \mathbf{T}_{(\mathbf{m})}\,dt$$

$$= [\mathbf{C}_\mathbf{S}^T + \frac{1}{2}\mathbf{C}_\mathbf{T}^T] \int \mathbf{S}_{(\mathbf{m})}\,dt$$

$$\iint t\,dt = \mathbf{C}_\mathbf{S}^T \iint \mathbf{S}_{(\mathbf{m})}\,dt + \mathbf{C}_\mathbf{T}^T \iint \mathbf{T}_{(\mathbf{m})}\,dt$$

$$= [\mathbf{C}_\mathbf{S}^T + \frac{1}{2}\mathbf{C}_\mathbf{T}^T]\,\mathbf{P}_{(\mathbf{m})} \int \mathbf{S}_{(\mathbf{m})}\,dt$$

Putting these results in (11), we get

$$f(t) \approx \left[\mathbf{C_S^T} + \frac{1}{2}\mathbf{C_T^T}\right]\left[\mathbf{P_{(m)}} + \mathbf{I_{(m)}}\right]\mathbf{P1ss_{(m)}}\mathbf{S_{(m)}} +$$

$$\left[\mathbf{C_S^T} + \frac{1}{2}\mathbf{C_T^T}\right]\left[\mathbf{P_{(m)}} + \mathbf{I_{(m)}}\right]\mathbf{P1st_{(m)}}\mathbf{T_{(m)}} \qquad (14)$$

$$\square\ \mathbf{D_{1S}^T}\,\mathbf{S_{(m)}} + \mathbf{D_{1T}^T}\mathbf{T_{(m)}}$$

From the two vectors $\mathbf{D_{1S}^T}$ and $\mathbf{D_{1T}^T}$ the samples of f(t) may be computed easily and compared with the exact solution vectors $\mathbf{D_S^T}$ and $\mathbf{D_T^T}$.

B. By the use of HF domain one-shot operational matrices

The one-shot operational matrices from equations (8) and (10), we can express RHS of (11) as

$$f(t) \approx [\,\mathbf{C_S^T}\ \mathbf{P1ss_{(m)}} + \mathbf{C_T^T}\ \frac{\mathbf{P1ss_{(m)}}}{2}\,]\,\mathbf{S_{(m)}} +$$

$$[\,\mathbf{C_S^T}\ \mathbf{P1st_{(m)}} + \mathbf{C_T^T}\ \frac{\mathbf{P1st_{(m)}}}{2}\,]\,\mathbf{T_{(m)}}$$

$$+ [\,\mathbf{C_S^T}\ \mathbf{P2ss_{(m)}} + \mathbf{C_T^T}\ \mathbf{P2ts_{(m)}}\,]\,\mathbf{S_{(m)}} +$$

$$[\,\mathbf{C_S^T}\ \mathbf{P2st_{(m)}} + \mathbf{C_T^T}\ \mathbf{P2tt_{(m)}}\,]\,\mathbf{T_{(m)}}$$

$$\square\ \mathbf{D_{2S}^T}\,\mathbf{S_{(m)}} + \mathbf{D_{2T}^T}\mathbf{T_{(m)}} \qquad (15)$$

Rearranging coefficients of $\mathbf{S_{(m)}}$, we have

$$\mathbf{D_{2S}^T} = [\,\mathbf{C_S^T}\ \mathbf{P1ss_{(m)}} + \mathbf{C_T^T}\ \frac{\mathbf{P1ss_{(m)}}}{2}\,] + [\,\mathbf{C_S^T}\ \mathbf{P2ss_{(m)}} + \mathbf{C_T^T}\ \mathbf{P2ts_{(m)}}\,]$$

Rearranging coefficients of $\mathbf{T_{(m)}}$, we get

$$\mathbf{D_{2T}^T} = [\,\mathbf{C_S^T}\ \mathbf{P1st_{(m)}} + \mathbf{C_T^T}\ \frac{\mathbf{P1st_{(m)}}}{2}\,] + [\,\mathbf{C_S^T}\ \mathbf{P2st_{(m)}} + \mathbf{C_T^T}\ \mathbf{P2tt_{(m)}}\,]$$

From the two vectors $\mathbf{D_{2S}^T}$ and $\mathbf{D_{2T}^T}$ the samples of f(t) may be computed.

After computation of f(t) by the above three methods using equations (12), (14) and (15)], we get the solution for the coefficients $\mathbf{D_S^T}, \mathbf{D_T^T}, \mathbf{D_{1S}^T}, \mathbf{D_{1T}^T}, \mathbf{D_{2S}^T}, \mathbf{D_{2T}^T}$ and can easily find out the different sets of samples which are compared in Table 1.

The percentage error columns of Table 1 show that the use of one-shot operational matrices are many shades better than using the first order integration matrices only. This is also supported by Fig. 4.

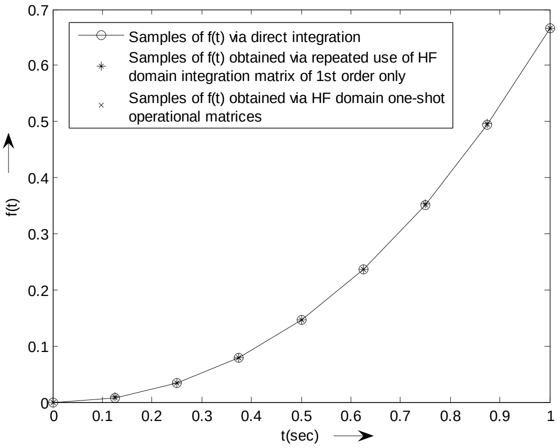

Fig 4 : Comparisons of three sets of samples of the function f(t) : (i) obtained via direct expansion in HF domain, (ii) obtained via repeated application of HF domain integration operational matrices of 1st order only and (iii) obtained via HF domain one-shot operational matrices of 2nd order.

VII TWO THEOREMS

It should be noted that all the operational matrices **P**, **P1ss**, **P1st**, **P1ts**, **P1tt**, **P2ss**,**P2st**, **P2ts**,

P2tt are of regular upper triangular nature and may be represented by ***S*** having the following general form :

$$\mathbf{S} = \sum_{0}^{j} a_n\,\mathbf{Q}^n$$

where, the delay matrix ***Q* [12]** is given by

$$\mathbf{Q_{(m)}}\ \square\ [\,0\quad 1\quad 0\quad 0\ \ldots\ \ldots\ \ldots\ 0\,]$$

We present the following two theorems regarding commutative property of matrices of class ***S*** and its polynomials.

Theorem 1:

If a regular upper triangular matrix ***S*** *of order m can be expressed as*

$$\boldsymbol{S} = \sum_{0}^{j} a_n \boldsymbol{Q}^n$$

where, the coefficients a_n*'s are constants,* $j \leq (m\text{-}1)$*, then the product of two matrices* ***S1*** *and* ***S2,*** *similar to* ***S,*** *raised to different integral powers p and q, is always commutative and of the form*

$$\boldsymbol{S1}^p \boldsymbol{S2}^q = \sum_{0}^{k} c_n \boldsymbol{Q}^n$$

where, the coefficients c_n*'s are constants , p, q, k are positive integers and* $k \leq (m\text{-}1)$*.*

Proof:

Let, $\mathbf{S1} = \sum_{0}^{l} a_n \mathbf{Q}^n$ and $\mathbf{S2} = \sum_{0}^{s} b_n \mathbf{Q}^n$

where l, s $\leq$ (m-1) and a_n and b_n are constant coefficients.

Then the product $\left[\mathbf{S1}^p \mathbf{S2}^q\right]$ is given by

$$\mathbf{S1}^p \mathbf{S2}^q = \left[\sum_{0}^{l} a_n \mathbf{Q}^n\right]^p \left[\sum_{0}^{s} b_n \mathbf{Q}^n\right]^q \quad (16)$$

The resulting polynomial would contain different coefficients with different powers of ***Q*** from 0 to u (say) where u $\leq$ (m-1), as ***Q*** has the property [12]

$$\mathbf{Q}^n_{(\mathbf{m})} = \mathbf{0}_{(\mathbf{m})}$$

for n > (m-1)

Then equation (16) reduces to

$$\mathbf{S1}^p \mathbf{S2}^q = \sum_{0}^{k} c_n \mathbf{Q}^n$$

for k $\leq$ (m-1) □

Theorem 2:

If a regular upper triangular matrix ***S*** *of order m can be expressed as*

$$\boldsymbol{S} = \sum_{0}^{v} a_n \boldsymbol{Q}^n$$

where, the coefficients a_n*'s are constants and* $v \leq (m\text{-}1)$ *then any polynomial of* ***S*** *can be expressed as*

$$\sum_{0}^{j} c_n \boldsymbol{S}^n = \sum_{0}^{k} d_n \boldsymbol{Q}^n$$

where, c_n*'s,* d_n*'s are constants and* $j, k \leq (m\text{-}1)$*.*

Proof:

The (r+1)th term of the polynomial $\sum_{0}^{j} c_n \mathbf{S}^n$ is

$$c_r \mathbf{S}^r = c_r \left[\sum_{0}^{v} a_n \mathbf{Q}^n\right]^r = c_r \sum_{0}^{w} f_n \mathbf{Q}^n = \sum_{0}^{w} g_n \mathbf{Q}^n \quad (17)$$

where w $\leq$ (m-1).

Since **Q** has the property

$$\mathbf{Q}^n_{(\mathbf{m})} = \mathbf{0}_{(\mathbf{m})} \qquad \text{for } n > (m-1)$$

Hence, putting r = n in equation (17), we can write as

$$\sum_{0}^{j} c_n \mathbf{S}^n = \sum_{0}^{j}\sum_{0}^{w} g_n \mathbf{Q}^n = \sum_{0}^{k} d_n \mathbf{Q}^n$$

□

VIII. SOLUTION OF SECOND ORDER DIFFERENTIAL EQUATION

In the following we present two methods based upon

(i) The repeated use of first order integration matrices.

(ii) The use of first and second order one-shot integration matrices.

A. Solution of a second order linear non-homogeneous differential equation via first order integration matrices **P1ss**, **P1st**, **P1ts** and **P1tt**

Consider the second order linear non-homogeneous differential equation

$$\ddot{x}(t) + a\,\dot{x}(t) + b\,x(t) = c \qquad (18)$$

where, a, b and c are positive constants.

Let, the initial conditions be $\dot{x}(0) = k2$ and $x(0) = k1$

The exact solution is $x(t) = m_1 e^{-pt} - m_2 e^{-qt} + m_3$

$$\text{where } p = q = \frac{a \pm \sqrt{\left(a^2 - 4b\right)}}{2}$$

and

$$\text{and } m_1 = \frac{\left[q\left(k1 - \frac{c}{b}\right) + k2\right]}{(q-p)},$$

$$m_2 = \frac{\left[p\left(k1 - \frac{c}{b}\right) + k2\right]}{(q-p)} \text{ and } m_3 = \frac{c}{b}$$

Integrating equation (18) twice we get,

$$\text{or, } x(t) + a\int x(t)\,dt + b\iint x(t)\,dt = c\iint u(t)\,dt + (k2+ak1)\int u(t)\,dt + k1\,u(t) \qquad (19)$$

Let $(k2 + ak1) = r1$ and $k1 = r2$

So, equation (19) takes the form

$$x(t) + a\int x(t)\,dt + b\iint x(t)\,dt = c\iint u(t)\,dt + r1\int u(t)\,dt + r2\,u(t) \qquad (20)$$

$$= c\mathbf{U}^T \iint \mathbf{S}_{(m)}\,dt + r1\mathbf{U}^T \int \mathbf{S}_{(m)}\,dt + r2\mathbf{U}^T \mathbf{S}_{(m)} \qquad (21)$$

Expanding all the time functions in m-term HF domain, we have

$$\mathbf{C}_\mathbf{S}^T\mathbf{S}_{(m)} + \mathbf{C}_\mathbf{T}^T\mathbf{T}_{(m)} + a[\mathbf{C}_\mathbf{S}^T \int \mathbf{S}_{(m)}\,dt + \mathbf{C}_\mathbf{T}^T \int \mathbf{T}_{(m)}\,dt] + b[\mathbf{C}_\mathbf{S}^T \iint \mathbf{S}_{(m)}\,dt + \mathbf{C}_\mathbf{T}^T \iint \mathbf{T}_{(m)}\,dt]$$

where, $x(t) \approx \mathbf{C}_\mathbf{S}^T\mathbf{S}_{(m)} + \mathbf{C}_\mathbf{T}^T\mathbf{T}_{(m)}$, $u(t) \approx \mathbf{U}^T\mathbf{S}_{(m)}$ and

$$\mathbf{U}^T = \underbrace{\begin{bmatrix} 1 & 1 & \dots & \dots & 1 & 1 \end{bmatrix}}_{m \text{ terms}}$$

Using (4), (5) and (6) and **P1ss**, **P1st**, **P1ts** and **P1tt**, we can express (21) as

$$\mathbf{C}_\mathbf{S}^T\mathbf{S}_{(m)} + \mathbf{C}_\mathbf{T}^T\mathbf{T}_{(m)} + 2a[\mathbf{C}_\mathbf{S}^T + \tfrac{1}{2}\mathbf{C}_\mathbf{T}^T] \int \mathbf{T}_{(m)}\,dt + 2b[\mathbf{C}_\mathbf{S}^T + \tfrac{1}{2}\mathbf{C}_\mathbf{T}^T]\,\mathbf{P}_{(m)} \int \mathbf{T}_{(m)}\,dt$$

$$= 2c\mathbf{U}^T\mathbf{P}_{(m)} \int \mathbf{T}_{(m)}\,dt + 2r1\mathbf{U}^T \int \mathbf{T}_{(m)}\,dt + r2\mathbf{U}^T\mathbf{S}_{(m)}$$

During following algebraic manipulations involving different upper triangular operational matrices and their polynomials, we recall their commutative properties as established by the two theorems presented in section VII.

Rearranging terms, we have

or, $(\mathbf{C_S^T} - r2\mathbf{U^T})\,\mathbf{S_{(m)}} + \mathbf{C_T^T}\mathbf{T_{(m)}} = 2\mathbf{U}^T[c\mathbf{P_{(m)}} + r1\mathbf{I_{(m)}}][\mathbf{P1ts_{(m)}}\mathbf{S_{(m)}} + \mathbf{P1tt_{(m)}}\mathbf{T_{(m)}}]$

$$- 2\,[\mathbf{C_S^T} + \frac{1}{2}\mathbf{C_T^T}][b\mathbf{P_{(m)}} + a\mathbf{I_{(m)}}][\mathbf{P1ts_{(m)}}\mathbf{S_{(m)}} + \mathbf{P1tt_{(m)}}\mathbf{T_{(m)}}] \quad (22)$$

Equating the coefficients of $\mathbf{S_{(m)}}$ from both sides

$$(\mathbf{C_S^T} - r2\mathbf{U}^T) = -2[\mathbf{C_S^T} + \frac{1}{2}\mathbf{C_T^T}][b\mathbf{P_{(m)}} + a\mathbf{I_{(m)}}]\,\mathbf{P1ts_{(m)}} + 2\mathbf{U}^T[c\mathbf{P_{(m)}} + r1\mathbf{I_{(m)}}]\,\mathbf{P1ts_{(m)}}$$

Let, $\mathbf{L} \;\square\; [-b\mathbf{P_{(m)}} - a\mathbf{I_{(m)}}]$ and $\mathbf{M} \;\square\; [c\mathbf{P_{(m)}} + r1\mathbf{I_{(m)}}]$

Then $\mathbf{C_S^T} = 2\,\mathbf{C_S^T}\,\mathbf{L}\,\mathbf{P1ts_{(m)}} + \mathbf{C_T^T}\mathbf{L}\,\mathbf{P1ts_{(m)}} + 2\mathbf{U}^T\mathbf{M}\,\mathbf{P1ts_{(m)}} + r2\mathbf{U}^T$

$$\mathbf{C_S^T} - r2\mathbf{U}^T = [2\,\mathbf{C_S^T}\,\mathbf{L} + \mathbf{C_T^T}\mathbf{L} + 2\mathbf{U}^T\mathbf{M}]\,\mathbf{P1ts_{(m)}} \quad (23)$$

Now rearranging the coefficients of $\mathbf{T_{(m)}}$ of equation (22), we get

$$\mathbf{C_T^T} = -2[\mathbf{C_S^T} + \frac{1}{2}\mathbf{C_T^T}][b\mathbf{P_{(m)}} + a\mathbf{I_{(m)}}]\,\mathbf{P1tt_{(m)}} + 2\mathbf{U}^T[c\mathbf{P_{(m)}} + r1\mathbf{I_{(m)}}]\,\mathbf{P1tt_{(m)}}$$

or, $$\mathbf{C_T^T} = [2\,\mathbf{C_S^T}\,\mathbf{L} + \mathbf{C_T^T}\mathbf{L} + 2\mathbf{U}^T\mathbf{M}]\,\mathbf{P1tt_{(m)}} \quad (24)$$

From (23) and (24)

$$(\mathbf{C_S^T} - r2\mathbf{U}^T) = \mathbf{C_T^T}\,\mathbf{P1tt_{(m)}}^{-1}\,\mathbf{P1ts_{(m)}}$$

Putting $\mathbf{P1tt_{(m)}}$, we get

$$(\mathbf{C_S^T} - r2\mathbf{U}^T) = \frac{2}{h}\,\mathbf{C_T^T}\,\mathbf{P1ts_{(m)}} \quad (25)$$

Solving simultaneous equations (23) and (25) for $\mathbf{C_S^T}$ and $\mathbf{C_T^T}$, we get

$$\mathbf{C_T^T} = 2\mathbf{U}^T[r2\mathbf{L} + \mathbf{M}][\frac{2}{h}\mathbf{I_{(m)}} - \frac{4}{h}\,\mathbf{P1ts_{(m)}}\,\mathbf{L} - \mathbf{L}]^{-1} \quad (26)$$

Putting the value of $\mathbf{C_T^T}$ from (26) in (25) we get

$$\mathbf{C_S^T} = \frac{4}{h}\mathbf{U}^T[r2\mathbf{L} + \mathbf{M}][\frac{2}{h}\mathbf{I_{(m)}} - \frac{4}{h}\,\mathbf{P1ts_{(m)}}\,\mathbf{L} - \mathbf{L}]^{-1}\,\mathbf{P1ts_{(m)}} + r2\mathbf{U}^T \quad (27)$$

For solving homogeneous equation we put c = 0 in equations (26) and (27) and can compute $\mathbf{C_S^T}$ and $\mathbf{C_T^T}$.

B. Solution of a second order linear non-homogeneous differential equation using one-shot operational matrices for integration of first and second orders

We consider equation (18) and use one-shot operational matrices for second order integration to determine its solution.

Proceeding as before we start from equation (20). We expand all the time functions in m-term HF domain and employ equations (8) and (10) for different one-shot operational matrices of orders 2 .

Thus we have,

$$\mathbf{C_S^T}\,\mathbf{S_{(m)}} + \mathbf{C_T^T}\mathbf{T_{(m)}} + a\,[\mathbf{C_S^T}\,\mathbf{P1ss_{(m)}} + \mathbf{C_T^T}\,\frac{\mathbf{P1ss_{(m)}}}{2}]\,\mathbf{S_{(m)}} + a[\mathbf{C_S^T}\,\mathbf{P1st_{(m)}} + \mathbf{C_T^T}\,\frac{\mathbf{P1st_{(m)}}}{2}]\,\mathbf{T_{(m)}}$$

$$+ b[\mathbf{C_S^T}\,\mathbf{P2ss_{(m)}} + \mathbf{C_T^T}\,\mathbf{P2ts_{(m)}}]\,\mathbf{S_{(m)}} + b[\mathbf{C_S^T}\,\mathbf{P2st_{(m)}} + \mathbf{C_T^T}\,\mathbf{P2tt_{(m)}}]\,\mathbf{T_{(m)}}$$

$$= \mathbf{U}^T[c\,\mathbf{P2ss_{(m)}}\,\mathbf{S_{(m)}} + r1\,\mathbf{P1ss_{(m)}}\,\mathbf{S_{(m)}} + c\,\mathbf{P2st_{(m)}}\,\mathbf{T_{(m)}} + r1\,\mathbf{P1st_{(m)}}\,\mathbf{T_{(m)}} + r2\,\mathbf{S_{(m)}}]$$

Rearranging the coefficients of $\mathbf{S_{(m)}}$, we have

$$\mathbf{C_S^T} + a\mathbf{C_S^T}\,\mathbf{P1ss_{(m)}} + a\,\mathbf{C_T^T}\frac{\mathbf{P1ss_{(m)}}}{2} + b\mathbf{C_S^T}\,\mathbf{P2ss_{(m)}} + b\,\mathbf{C_T^T}\,\mathbf{P2ts_{(m)}} = \mathbf{U}^T[c\,\mathbf{P2ss_{(m)}} + r1\,\mathbf{P1ss_{(m)}} + r2\mathbf{I_{(m)}}]$$

or, $$\mathbf{C_S^T}[\mathbf{I_{(m)}} + a\,\mathbf{P1ss_{(m)}} + b\,\mathbf{P2ss_{(m)}}] + \mathbf{C_T^T}[a\frac{\mathbf{P1ss_{(m)}}}{2} + b\,\mathbf{P2ts_{(m)}}] = \mathbf{U}^T[c\,\mathbf{P2ss_{(m)}} + r1\,\mathbf{P1ss_{(m)}} + r2\mathbf{I_{(m)}}] \quad (28)$$

Rearranging the coefficients of $\mathbf{T_{(m)}}$, we get

$$\mathbf{C_T^T} + a\mathbf{C_S^T}\,\mathbf{P1st_{(m)}} + a\mathbf{C_T^T}\frac{\mathbf{P1st_{(m)}}}{2} + b\mathbf{C_S^T}\,\mathbf{P2st_{(m)}} + b\,\mathbf{C_T^T}\,\mathbf{P2tt_{(m)}} = \mathbf{U}^T[c\,\mathbf{P2st_{(m)}} + r1\,\mathbf{P1st_{(m)}}]$$

or, $$\mathbf{C_S^T}[a\,\mathbf{P1st_{(m)}} + b\,\mathbf{P2st_{(m)}}] + \mathbf{C_T^T}[\mathbf{I_{(m)}} + a\frac{\mathbf{P1st_{(m)}}}{2} + b\,\mathbf{P2tt_{(m)}}] = \mathbf{U}^T[c\,\mathbf{P2st_{(m)}} + r1\,\mathbf{P1st_{(m)}}] \quad (29)$$

In equation (28), let us define

$$\mathbf{X} \equiv [\mathbf{I_{(m)}} + a\,\mathbf{P1ss_{(m)}} + b\,\mathbf{P2ss_{(m)}}] \text{ and}$$

$$\mathbf{Y} \equiv [a\frac{\mathbf{P1ss_{(m)}}}{2} + b\,\mathbf{P2ts_{(m)}}]$$

Then (28) may be written as

$$\mathbf{C_S^T}\,\mathbf{X} + \mathbf{C_T^T}\,\mathbf{Y} = \mathbf{U}^T[c\,\mathbf{P2ss_{(m)}} + r1\,\mathbf{P1ss_{(m)}} + r2\mathbf{I_{(m)}}] \quad (30)$$

In equation (29), let

$$\mathbf{W} \equiv [a\,\mathbf{P1st_{(m)}} + b\,\mathbf{P2st_{(m)}}] \text{ and } \mathbf{Z} \equiv [\mathbf{I_{(m)}} + a\frac{\mathbf{P1st_{(m)}}}{2} + b\,\mathbf{P2tt_{(m)}}]$$

Then equation (29), may be expressed as

$$\mathbf{C_S^T}\,\mathbf{W} + \mathbf{C_T^T}\,\mathbf{Z} = \mathbf{U}^T[c\,\mathbf{P2st_{(m)}} + r1\,\mathbf{P1st_{(m)}}] \quad (31)$$

Solving the matrix equations (30) and (31) for $\mathbf{C_S^T}$ and $\mathbf{C_T^T}$, we get

$$\mathbf{C_T^T}[\mathbf{YX}^{-1} - \mathbf{ZW}^{-1}] = \mathbf{U}^T[c\,\mathbf{P2ss_{(m)}} + r1\,\mathbf{P1ss_{(m)}} + r2\mathbf{I_{(m)}}]\mathbf{X}^{-1} - \mathbf{U}^T[c\,\mathbf{P2st_{(m)}} + r1\,\mathbf{P1st_{(m)}}]\mathbf{W}^{-1} \quad (32)$$

Let, $[\mathbf{YX}^{-1} - \mathbf{ZW}^{-1}] = \mathbf{M1}$

and $\mathbf{U}^T[c\,\mathbf{P2ss_{(m)}} + r1\,\mathbf{P1ss_{(m)}} + r2\mathbf{I_{(m)}}]\,\mathbf{X}^{-1} - \mathbf{U}^T[c\,\mathbf{P2st_{(m)}} + r1\,\mathbf{P1st_{(m)}}]\mathbf{W}^{-1} = \mathbf{M2}$

So, equation (32), becomes,

$$\mathbf{C_T^T}\,\mathbf{M1} = \mathbf{M2}$$

or, $$\mathbf{C_T^T} = \mathbf{M2M1}^{-1} \quad (33)$$

Putting in equation (31),

$$\mathbf{C_S^T} + \mathbf{M2M1}^{-1}\mathbf{ZW}^{-1} = \mathbf{U}^T[c\,\mathbf{P2st_{(m)}} + r1\,\mathbf{P1st_{(m)}}]\mathbf{W}^{-1}$$

Let, $\mathbf{M2M1}^{-1}\mathbf{ZW}^{-1} = \mathbf{M3}$ and $\mathbf{U}^T[c\,\mathbf{P2st_{(m)}} + r1\,\mathbf{P1st_{(m)}}]\mathbf{W}^{-1} = \mathbf{M4}$

Then,

$$\mathbf{C_S^T} + \mathbf{M3} = \mathbf{M4}$$

or, $$\mathbf{C_S^T} = \mathbf{M4} - \mathbf{M3} \quad (34)$$

For solving homogeneous equation we put c = 0 in equations (32) and (34) and can compute $\mathbf{C_S^T}$ and $\mathbf{C_T^T}$.

It is known that inversion of upper or lower triangular matrices can be performed by simple decomposition and multiplication [13]. Hence the inversions in equations (26), (279), (33), (34) will not pose any computational burden while solving for the HF domain solution matrices $\mathbf{C_S^T}$ and $\mathbf{C_T^T}$.

IX. ILLUSTRATIVE EXAMPLE 2

Let us now treat an example to show the effective application of the first order integration matrices $\mathbf{P1ss}$, $\mathbf{P1st}$, $\mathbf{P1ts}$ and $\mathbf{P1tt}$ for solving a second order linear differential equation and compare the result with the solution obtained via the use of one-shot integration operational matrices of second order.

Consider the equation

$$\ddot{x}(t) + 6\,\dot{x}(t) + 8\,x(t) = 8$$

with $\dot{x}(0) = -2$, $x(0) = 3$

The solution is $x(t) = 3\exp^{-2t} - \exp^{-4t} + 1$

The HF domain vectors obtained from the direct expansion of x(t) are

C_S = [3.0000 2.7299 2.4517 2.1940 1.9683 1.7774 1.6196 1.4911]

C_T = [-0.2701 -0.2782 -0.2577 -0.2257 -0.1909 -0.1578 -0.1285 -0.1034]

Using equations (26) and (27), first order integration matrices $\mathbf{P1ss}$, $\mathbf{P1st}$, $\mathbf{P1ts}$ and $\mathbf{P1tt}$ with m = 8, the solution of the differential equation yields x(t) as

C_S = [3.0000 2.7333 2.4548 2.1955 1.9683 1.7761 1.6175 1.4886]

C_T = [-0.2667 -0.2785 -0.2593 -0.2273 -0.1921 -0.1586 -0.1289 -0.1036]

However, using equations (33) and (34), when we use one-shot operational matrices for integration of 1st and 2nd orders, the solution of the differential equation yields the samples of x(t) as

C_S = [3.0000 2.7313 2.4520 2.1927 1.9659 1.7745 1.6166 1.4883]
C_T = [-0.2687 -0.2793 -0.2593 -0.2268 -0.1914 -0.1579 -0.1283 -0.1031]

TABLE II:
COMPARISON OF RESULTS VIA TWO APPROACHES WITH THE EXACT SOLUTION

t (sec)	Exact samples of x(t)	Samples of x(t) via HF domain integration matrices of 1st order	% Error	Samples of x(t) via HF domain one-shot integration matrices of 2nd order	% Error
0	3.0000	3.0000	0	3.0000	0
$\frac{1}{8}$	2.7299	2.7333	-0.1245	2.7313	-0.0513
$\frac{2}{8}$	2.4517	2.4548	-0.1264	2.4520	-0.0122
$\frac{3}{8}$	2.1940	2.1955	-0.0684	2.1927	0.0593
$\frac{4}{8}$	1.9683	1.9683	0	1.9659	0.1219
$\frac{5}{8}$	1.7774	1.7761	0.0731	1.7745	0.1632
$\frac{6}{8}$	1.6196	1.6175	0.1297	1.6166	0.1852
$\frac{7}{8}$	1.4911	1.4886	0.1677	1.4883	0.1878
1	1.3877	1.385	0.1945	1.3852	0.1802

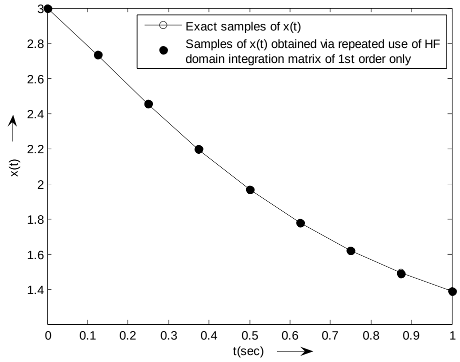

Fig 5: Comparison of actual samples of the function x(t) with the samples obtained via repeated application of HF domain first order integration operational matrices $\mathbf{P1ss}$, $\mathbf{P1st}$, $\mathbf{P1ts}$ and $\mathbf{P1tt}$.

Table 2 shows the comparison of results via two approaches with the exact solution. Although the use of one-shot operational matrices of 1st and 2nd orders incur less error, Table 2 brings out the result where both 1st order repeated integration matrices and one-shot operational matrices compete closely for the approximation of the exact solution for this specific case, as apparent from Figs. 5 and 6. This is possibly due to the large coefficients of $\dot{x}(t)$ and $x(t)$ compared to that of $\ddot{x}(t)$.

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TABLE III:
COMPARISON OF RESULTS VIA TWO APPROACHES WITH THE EXACT SOLUTION FOR M = 4, T= 0.4s.

t (sec)	Exact samples of x(t)	Samples of x(t) via HF domain integration matrices of 1st order	% Error	Samples of x(t) via HF domain one-shot integration matrices of 2nd order	% Error
0	2.0000	2.0000	**0**	2.0000	**0**
0.1	1.0806	1.2000	**-11.0494**	1.1429	**-5.7653**
0.2	-0.8323	-0.5600	**0.3272**	-0.6939	**16.6286**
0.3	-1.9800	-1.8720	**5.4545**	-1.9359	**2.2272**
0.4	-1.3073	-1.6864	**-28.9987**	-1.5185	**-16.1554**

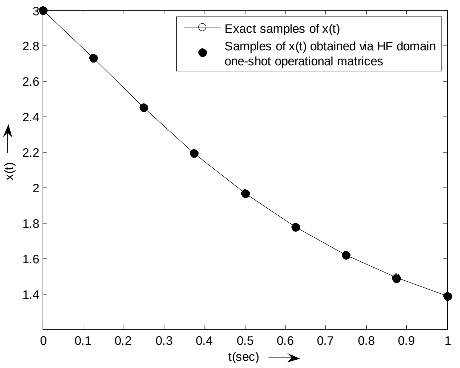

Fig 6 : Comparison of actual samples of the function x(t) with the solution of samples obtained via HF domain one-shot integration operational matrices of 1st and 2nd orders.

Illustrative Example 3

Consider the equation

$$\ddot{x}(t) + 100\,x(t) = 0$$

with $\dot{x}(0) = 0$, $x(0) = 2$

The solution is x(t) = 2cos10t.

TABLE IV:
COMPARISON OF RESULTS VIA TWO APPROACHES WITH THE

t (sec)	Exact samples of x(t)	Samples of x(t) via 4th order Runge Kutta method	% Error	Samples of x(t) via HF domain one-shot integration matrices of 2nd order	% Error
0	2.0000	2.0000	0	2.0000	0
0.05	1.7552	1.7552	0	1.7600	-0.2735
0.1	1.0806	1.0812	-0.0555	1.0976	-1.5732
0.15	0.1415	0.1429	-0.9894	0.1718	-21.4134
0.2	-0.8323	-0.8302	0.2523	-0.7953	4.4455
0.25	-1.6023	-1.6000	0.1435	-1.5715	1.9222
0.3	-1.9800	-1.9783	0.0859	-1.9705	0.4797
0.35	-1.8729	-1.8727	0.0107	-1.8966	-1.2654
0.4	-1.3073	-1.3091	-0.1377	-1.3675	-4.6049

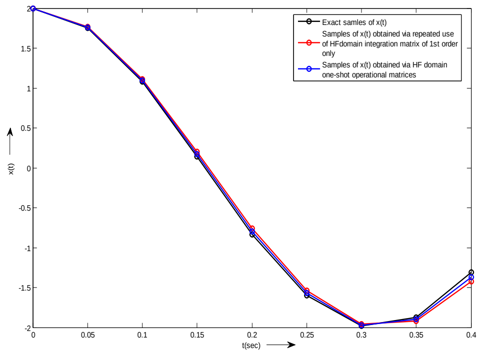

Fig 7 : Comparison of actual samples of the function x(t) with the (i) solution of samples obtained via repeated use of HF domain first order integration operational matrices and (ii) solution of samples obtained via HF domain one-shot operational matrices for integration of 1st and 2nd orders for m = 4, T = 0.4s.

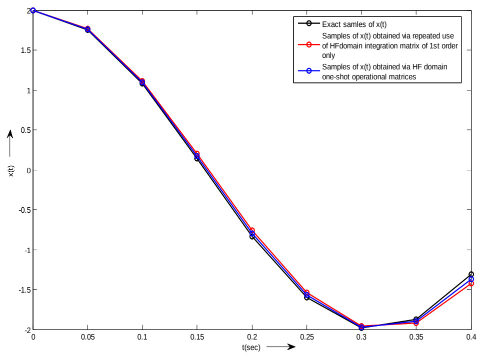

Fig 8 : Comparison of actual samples of the function x(t) with the (i) solution of samples obtained via repeated use of HF domain first order integration operational matrices and (ii) with the solution of samples obtained via HF domain one-shot operational matrices for integration of 1st and 2nd orders for m =8, T = 0.4s.

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TABLE V :
COMPARISON OF RESULTS VIA TWO APPROACHES WITH THE EXACT SOLUTION FOR M = 12, T = 0.4s

t (sec)	Exact samples of x(t)	Samples of x(t) via HF domain integration matrices of 1st order	% Error	**Samples of x(t) via HF domain one-shot integration matrices of 2nd order**	% Error
0	2.0000	2.0000	0	2.0000	0
0.03	1.8899	1.8919	-0.1058	1.8909	-0.0529
0.07	1.5718	1.5793	-0.4772	1.5755	-0.2354
0.1	1.0806	1.0959	-1.4159	1.0883	-0.7126
0.13	0.4705	0.4940	-4.9947	0.4823	0.0003
0.17	-0.1914	-0.1612	15.7785	-0.1763	7.8892
0.2	-0.8323	-0.7990	4.0009	-0.8156	2.0065
0.23	-1.3815	-1.3505	2.2439	-1.3660	1.1219
0.27	-1.7787	-1.7559	1.2818	-1.7674	0.6353
0.3	-1.9800	-1.9715	0.4293	-1.9759	0.2071
0.33	-1.9633	-1.9740	-0.5450	-1.9689	-0.2852
0.37	-1.7306	-1.7631	-1.8779	-1.7471	-0.9534
0.4	-1.3073	-1.3616	-4.1536	-1.3348	-2.1036

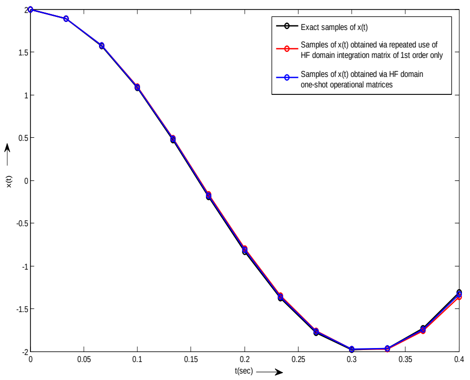

Fig 9 : Comparison of actual samples of the function x(t) with the (i) solution of samples obtained via repeated use of HF domain first order integration operational matrices and (ii) with the solution of samples obtained via HF domain one-shot operational matrices for integration of 1st and 2nd orders for m=12, T = 0.4s.

Example 3 establishes that one-shot operational matrices easily stand ahead in comparison to integration matrices of 1st order as the effect of $\ddot{x}(t)$ is poignant in this problem. The approximation gets better each time with increasing value of m. Fig. 8, 9, 10 and Tables 3, 4 and 5 support this with values of m chosen as 4, 8 and 12 respectively

Since inversion of upper or lower triangular matrices can be computed by simple decomposition and multiplication [13], the inversions in equations (26), (27), (33), (34)

X. Conclusion

In this paper, a new set of orthogonal functions, comprised of triangular functions and sample-and-hold functions, termed hybrid functions (HF) has been proposed. This new function set proves to be efficient for function approximation which is established via one illustrative example.

As in the case of Walsh and block pulse functions, the operational matrices for integration, namely **P1ss** , **P1st**, **P1ts** and **P1tt** , in hybrid function domain, are also derived. These matrices are employed for integration of a ramp function with reasonable error limits.

One shot integration operational matrices like **P2ss**, **P2st**, **P2ts**, **P2tt** for 2nd order repeated integration have been derived. An example is treated to compare the results obtained via repeated use of 1st order operational matrices and with the results obtained using second order one-shot operational matrices. These results are compared in Table 1 and Fig. 4.

Finally, to solve second order differential equations, two methods are proposed. One using only1st order matrices **P1ss** , **P1st**, **P1ts** and **P1tt** and the other employing one-shot operational matrices like **P2ss**, **P2st**, **P2ts**, **P2tt**. It is observed that the method based upon one-shot operational matrices produces much accurate result compared to the method using only 1st order matrices. Table 3, 4 and 5 compare the results and sample-wise errors in detail.

TABLE VI:

COMPARISON OF THREE SETS OF SAMPLES OF THE FUNCTION x(t) OF EXAMPLE 4 FOR M= 8 and T = 0.4 s : (i) OBTAINED VIA 4TH ORDER RUNGE-KUTTA METHOD, (ii) EXACT SOLUTION OF THE FUNCTION x(t) (iii) SOLUTION OBTAINED VIA HF DOMAIN ONE-SHOT OPERATIONAL MATRICES OF 2ND ORDER

t (sec)	Exact samples of x(t)	Samples of x(t) via 4th order Runge Kutta method	% Error	Samples of x(t) via HF domain one-shot integration matrices of 2nd order	% Error
0	2.0000	2.0000	0	2.0000	0
0.05	1.7552	1.7552	0	1.7600	-0.2735
0.1	1.0806	1.0812	-0.0555	1.0976	-1.5732
0.15	0.1415	0.1429	-0.9894	0.1718	-21.4134
0.2	-0.8323	-0.8302	0.2523	-0.7953	4.4455
0.25	-1.6023	-1.6000	0.1435	-1.5715	1.9222
0.3	-1.9800	-1.9783	0.0859	-1.9705	0.4797
0.35	-1.8729	-1.8727	0.0107	-1.8966	-1.2654
0.4	-1.3073	-1.3091	-0.1377	-1.3675	-4.6049

The same second order differential equation is solved via the well established classical 4th order Runge-Kutta method and the results obtained via HF domain one-shot operational matrices are compared in Fig. 10 and Table 6. It is noted that HF domain based analysis competes closely with the classical 4th order Runge-Kutta method which is somewhat better. The important fact is, the HF domain technique can (i) approximate square integrable time functions (ii) integrate time functions and (iii) solve higher order differential equations with considerable accuracy.
Since inversion of upper or lower triangular matrices can be computed by simple decomposition and multiplication [13], the inversions in equations (26), (27), (33), (34) will not pose any additional computational burden.

Finally, another advantage of the HF based analysis is, the sample-and-hold function based results may easily be obtained by simply dropping the triangular part of the hybrid function domain solutions.

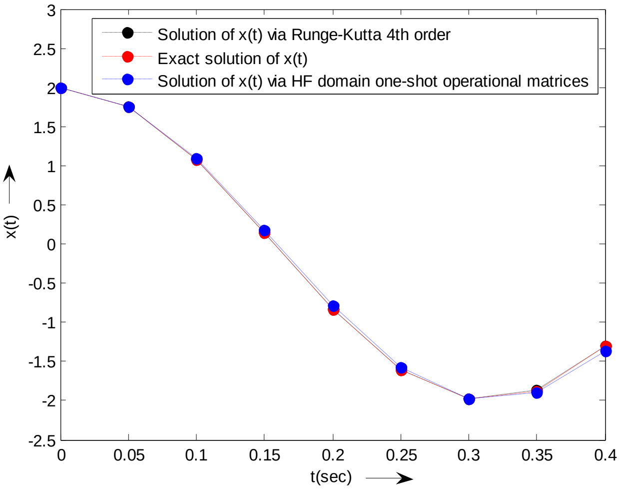

Fig 10: Comparisons of three sets of samples of the function x(t) : (i) obtained via 4th order Runge-Kutta method, (ii) exact solution of the function x(t) (iii) solution obtained via HF domain one-shot operational matrices of 2nd order of Example 4 with m = 8, T = 0.4s.

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