International Journal of Scientific Engineering and Technology (ISSN : 2277-1581)
www.ijset.com, Volume No.1, Issue No.4, pg : 39-41 01 Oct. 2012

REPRESENTATION OF NUMBERS

Sundaram Ramchandran
SUNY Binghamton, Binghamton, NY, Aug 2001
Email: sundarramchandran@hotmail.com, zenkalidas@yahoo.com

Abstract

For modulating signals, the compact (see note 1) representation of the signals assumes importance to simplify the design of circuits and in the context of design of systems that are robust This article examines the nature of numbers and goes on to survey the methods of representing algebraic irrational numbers. It also briefly examines the methods of physically realizing the same, especially in the context of frequency modulation techniques , speech processing and other applications as also to issues of sampling in DSP.

Keywords
Irrational, Transcendental, Polynomial, Roots, Phi, imaginary, complex, recursive, dynamical system, chaos, hysteresis, base, fields, rings, algebraic structure.

INTRODUCTION

At the risk of stating the obvious, real numbers can be divided into the following categories:

- Natural numbers / Integers
- Rational Numbers
- Irrational Algebraic numbers
- Irrational Non-algebraic Transcendental Numbers

Natural numbers / Integers are of course easily represented. Of course, there is always the issue of compact representation as well as the largest integer that can be represented in a given system. Some methods that have been adopted for handling this are: (Other than as a string of characters)

- The floating point / exponential representation (exponent mantissa)
- As a polynomial to the base of a large number
- In terms of its prime factors

Rational numbers can be represented as an ordered pair of numbers with the basic arithmetical operations represented appropriately (The more mathematically inclined may want to investigate and prove / disprove the nature of rational numbers as a field). One interesting idea could be to map the above to the complex plane after embedding it in the complex number system (defined later)

Irrational numbers are basically of two types:

- Algebraic numbers (Those that can be represented as the roots of polynomials with rational coefficients). This is typically of the form of roots of integers and their sums, products etc.
- Transcendental numbers such as pi , e that cannot be so represented (Of course they can be represented through transcendental functions such as trigonometric functions and / or differential equations but that is another story)

There have also been mathematical definitions of rational numbers as limits of sequences both irrational algebraic and non-algebraic or transcendental numbers and of integers as limits of rational numbers. Of course integers form a subset of the group of rational numbers with Prime Integers forming a subset of the integers.

Real numbers again form a subset of the set of complex numbers. Interestingly, according to the ***Fundamental Theorem of Algebra***, the set of complex numbers is complete in the sense that any nth degree polynomial with complex coefficients will have n roots (including multiple roots) (which, as we have seen, is not the case with polynomials with integral / rational or real coefficients). Those who want to know more about complex numbers can go through the Appendix. The more mathematically inclined may like to investigate ***Cantor's*** theories relating to the cardinality of the set of real numbers, the set of rational numbers (including integers as a subset including the primes as a sub-sub-set) and the group of complex numbers.

Prime polynomials over a field are those polynomials that are irreducible and do not have roots in that field. For example, if a polynomial has only imaginary roots, it is considered prime over the real field. Interestingly, there seems to be a dual relationship between transcendental numbers which cannot be expressed as roots of polynomials with real coefficients and prime polynomials over the real field though their cardinalities (sizes) are at opposite ends of the spectrum. In this section, we will cover the representation of algebraic irrational numbers.

METHODS

The compact representation of irrational numbers assumes importance especially in signal processing applications such as signal modulation by carrier signals (for example generation of sinusoidal carrier modulation, especially those with frequencies that are pairwise mutually incommensurate (not rationally related to one another)). Accurate representations of such numbers beyond those provided by current methods may be necessary to prevent instabilities especially in applications and systems that are chaotic or extremely sensitive to initial conditions. One of the simplest methods of representing irrational square roots would be to represent them as the distance / magnitude of a vector in the plane (links to pythagoras theorem) with possible conditions on the phase (angle between the coordinates) (Also known as the ladder method). But this would only be valid for square roots and not for nth roots in general.

One of the methods that have been adopted to approximate irrational numbers is that of a dynamical system / process / algorithm (linked to continued fractions and related areas). For example, a recursive / iterative algorithm for computing square roots is as follows.Suppose the square root of alpha is to be computed. We choose the initial value x(1) which is greater than alpha and then define x(n+1) as :

x(n+1) = (x(n) + (alpha /x(n)) / 2

The more mathematically inclined may like to prove that the sequence decreases monotonically and find the rate of convergence and find better methods of accelerating convergence such as shanks methods. (This is taken from the book "Principles of mathematical analysis" by Walter Rudin -> Third Edition. Page 81 Chapter 3 -> Numerical Sequences and Series ex 16.) Similar methods (linked to continued fractions) have also been used to approximate transcendental numbers such as pi and e but these will be covered at a later point of time. **A more general and more mathematical method is to represent them as the zero(s) of a polynomial with integer coefficients (extended to rational coefficients).** (Again, the more mathematically inclined may like to investigate extensions to the above to fractional exponents (links to elliptic curves where y^2 is equated to a cubic in x on the right)) as also to application to problems involving nested roots / horner's polynomials.

One method of realizing the above electronically seems to be representing the polynomial through an array of delay elements with integral or rational weights (links to z transform or weighted powers of the time variable. Of course the system can be implemented in parallel) and have a system with the resultant equated / clamped to zero and with taps for extracting the different roots to avoid cycling between the different roots or being sensitive to initial conditions. To solve for repeated roots, one could possibly have phase locked loops which synchronize the various tap outputs and equalize them. Essentially, a general extension of the idea would be to have a kind of network which could possibly be reconfigured (linked to what is known as neural networks) where the processing units could realize various functions (which could be linked to other functions) and where the links could denote both input as well as operations such as addition / component wise multiplication, general polynomial multiplication (Linked to convolution of signals), correlation and others. This would allow us to represent variables and symbolic computation by linking them to the input. Function evaluation would proceed in forward while solving equations would start with the output and possibly involve feedback circuits in general. This kind of representation has the advantage that it can be extended to any kind of data including signals / images / 2ds and 3d shapes. (This is also linked to what is known as Waveform computing). This may also be related to / extended to realizing a matrix of functions. It may be possible to solve constrained optimization problems by implementing inequalities through bounds / limits set through stabilizing mechanisms.

One could sample a real interval fairly densely within a certain interval by choosing a fairly high degree polynomial and increasing / decreasing the value of the coefficients (with bounds on the height/norm of the polynomial). The ordering could be based on the powers of the coefficients.

Interestingly, this is also connected to the dense covering of the unit circle by the superposition of 2 oscillators moving on mutually orthogonal axes with mutually incommensurate frequencies (lissajous figures). If the orthogonality of these oscillators (linked to vector sum) could be represented by a pair of signals in quadrature like sine and cosine), these may provide an alternative (possibly simpler) way of representing and generating frequency modulation signals as compared to those involving imaginary phases which may involve damped exponentials. This could also be connected to the modeling of voice signals that are said to be quasi-periodic. In fact, for many mutually incommensurate values of the frequencies and appropriate weights, the resultant signal could possibly be one of a family of space filling curves. It would be interesting to consider extensions to quaternions. Again, the more mathematically inclined may like to investigate the embedding of rational numbers within the irrational numbers and vice-versa with links to approximation of integrals.

Those interested in physics may like to investigate connections of the above to circular polarization.

SOME INTERESTING RESULTS

The above method could also be used to represent **i** (the square root of -1), the basis of extension of the real number system to the complex plane as the root(s) of the polynomial:

X^2 + 1 = 0

Interestingly, the golden mean or **phi** (1+/- sqrt (5))/ 2 which occur(s) in various areas of aesthetics, geometry, architecture is / are defined as the roots of the polynomial:

X^2 = X + 1 or X^2 – X -1 = 0.

In fact, this simple relationship between scaling and addition / multiplication (In fact any power of phi can be represented as a multiple of phi + a constant (which would be a member of the Fibonacci series which starts with 1 1and 1 where every term is the sum of two preceding terms for which there is a simple formula for computing the nth term) which is valid for all powers is the key to simplification of computation, compact representations of numbers and their powers as well as implementing the shift register schemes as mentioned above. Especially since the smaller value of phi , i.e. (1-sqrt (5))/2, is smaller than 1, it may be possible in certain cases to realize computations of infinite series as sums of geometric progressions.

Geometrically, it occurs in a variety of packing, dissection and tiling problems and has the property that if a line (say of unit length) is divided in the golden mean say a/b where a is the smaller part, then by definition, b / 1 (or a + b) = a/b

The slow growth of powers of this number (This could be the simplest such irrational number since it is the root of a polynomial which has a combination of a low degree and low height in the sense of absolute value of coefficients. Similar roots of higher degree polynomials may create problems in the sense that not all roots may be pairwise incommensurate or small) may be important in a number of applications where fine tuning and regulation is important (as compared to speed of computation which may not be an issue nowadays) to prevent instabilities, overshooting etc (Of course Chebyshev polynomials are also applicable in this context but these may not be so easily computed.

One could also combine the two themes) such as control systems, regulating the volume of fluid in dispensing systems, controlling dosage of medicine which may be important not only because a higher dosage may be more toxic but also because there may be a nonlinear relationship between dosage and effect. The above may also be important in bounding a system (as mentioned earlier in the example of optimization problems) and slowing the process of saturation as also in better utilization of physical memory.

It would be interesting to investigate the applications / consequences of extending / complementing the natural / integral number system (and its extension the rationals) by including the golden mean , i.e., (1-sqrt(5))/2 and its relevance to multivalued / fuzzy logic. (For example, 1 = hi^2 – phi , 0 = phi – phi -1 = phi -phi ^2 and so on) with possible links to alternative bases for computing architectures (instead of the usual 0/1.

At a more theoretical level, one interesting theme is the link between multiple roots of a polynomial and non invertible systems with links to multi-stability and possibly chaos (as well as the link between Fourier transform and mappings from coefficient to root space and the reverse). The above could also be linked to chebyshev polynomials of irrational argument / order.

CONCLUSION

In this article we examined some methods of representing irrational numbers (especially algebraic irrational numbers) and some interesting results (especially those relating to Phi , the golden mean, the imaginary number i) and some possible applications to dynamical (possibly chaotic) systems etc (the latter will be investigated in subsequent articles).

Another interesting direction would be that of generalization of the above to arbitrary algebraic structures (for example polynomial rings over abstract fields) , such as defining a continuous group (Lie group ??) through discrete groups with possible links to Galois theory etc.

Also,. another interesting aspect that has been examined is the idea of the best physical representation of the essence of a symbolic and abstract algebraic structure.

Notes:1. In this document, compact representation is meant to mean the simplest representation in 2 senses, one in terms of the length of the representation in memory and secondly, the ease of understanding and the accuracy and the length of the representation for display purposes.

REFERENCES

1. **Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise Manfred Schroeder**
2. **Principles of Mathematical Analysis - Walter Rudin - 3rd Edition**
3. **Abstract Algebra - Herstein**
4. **S .Ramchandran and M.Chatterjee, "Nonlinear Dynamics of a Bragg cell under intensity feedback in the near-Bragg, four-order regime".Applied Optics / Vol 41, No 29, / October 2002**
5. **M.Chatterjee and S.Ramchandran, "Feedback correction of angular error in grating readout", SPIE Conference, San Diego, July 2001, vol. 4470, pp. 127-137**

APPENDIX

Complex Numbers

Complex numbers are of the form a+bi where both a and b are real (and relate to x and y coordinates in the plane) and i=sqrt (-1) as shown earlier in the document. An alternative representation of complex numbers is of the form r*e^ (i*theta) where r is the magnitude which is defined as sqrt (a^2+b^2) and theta is the phase which is defined by tan (theta) = b/a e is related to the exponential function and is also called the natural logarithm.