

# The $p$ -Quasihyponormality of the Generalized Aluthge Transform

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**Abstract :**

Let  $T = U|T|^p$  be the polar decomposition of  $p$ -Quasihyponormal operator for  $0 < p < 1$ . Then the operator  $|\tilde{T}_{s,t}| = |T|^s U |T|^t$ ,  $0 < s < 1, 0 < t < 1$  is  $\left(\frac{1}{2(s+t)}\right)$ -Quasihyponormal and the operator  $|\tilde{T}_{r,t}| = |T|^r U |T|^{t-r}$ ,  $0 < r, t < 1, r \geq t$  is  $q$ -Quasihyponormal where  $q = \max\{p+r+t, 2\}$ .

**Keywords:**  $p$ -Quasihyponormal operator, Aluthge Transform

**Mathematics Subject Classification:** 47B20, 47A63

**Introduction:**

Let  $H$  denotes a separable complex infinite dimensional Hilbert space and  $B(H)$  the algebra of all bounded linear operators on  $H$ . An operator  $T$  on a Hilbert space  $H$  is said to be  $p$ -hyponormal if  $(T^*T)^p \geq (TT^*)^p$  for a positive number  $p$  and if  $p = \frac{1}{2}$ , then  $T$  is semi-hyponormal. An operator  $T$  on a Hilbert space  $H$  is said to be  $p$ -quasihyponormal if  $T^* \left( (T^*T)^p - (TT^*)^p \right) T \geq 0$  for  $p > 0$ . If  $p = 1$ , then  $T$  is quasihyponormal and if  $p = \frac{1}{2}$ , then  $T$  is semi-quasihyponormal. Every  $p$ -quasihyponormal operator is a  $q$ -quasihyponormal operator for  $q \leq p$ . S.C .Arora and Pramod Arora [2] introduced  $p$ -quasihyponormal operator and they studied some properties of  $p$ -Quasihyponormal using the operator  $|\tilde{T}| = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ ,  $T = U|T|$  is the polar decomposition of  $T$ . MI Young Lee and Sang Hun Lee [3] studied various properties of  $p$ -Quasihyponormal for the operator  $|\tilde{T}_\varepsilon| = |T|^\varepsilon U |T|^{1-\varepsilon}$ ,  $0 < \varepsilon \leq \frac{1}{2}$ . Takashi Yoshino [6] defined more generally for any  $s$  and  $t$  such as  $s \geq 0$  and  $t \geq 0$   $\tilde{T}_{s,t} = |T|^s U |T|^t$  the  $p$ -hyponormality of the

Aluthge transform of  $T$ . Throughout this paper we denote  $R(T)$ , the range space of  $T$  and we consider  $0 < p < 1$ .

**Furuta's Inequality: [5]**

Let  $A$  and  $B$  be bounded self-adjoint operators such that  $A \geq B \geq 0$ . Then for each  $r \geq 0$ ,

$$\left( B^r A^u B^r \right)^{\frac{1}{q}} \geq B^{\left( \frac{u+2r}{q} \right)}$$

$$A^{\left( \frac{u+2r}{q} \right)} \geq \left( A^r B^u A^r \right)^{\frac{1}{q}}$$

for each  $u$  and  $q$  such that  $u \geq 0, q \geq 1$  and  $(1+2r)q \geq s+2r$ .

**Lemma: 1**

For  $T = U|T|^p$ ,  $R(\tilde{T}_{s,t}) \subset R(|T|^p)$ ,  $0 < s, t < 1$

**Proof:**

By [2, Theorem 2.1]  $R(\tilde{T}_{s,t}) \subset R(|T|^s)$

Since  $|T|^r$  is positive for all  $0 < r < 1$ , therefore we have  $R(\tilde{T}_{s,t}) \subset R(|T|^p)$ .

**Theorem: 1**

Let  $T = U|T|^p$  be the polar decomposition of a  $p$ -Quasihyponormal operator and  $U$  is unitary. Then  $\tilde{T}_{s,t} = |T|^s U |T|^t$  is  $\left(\frac{1}{2(s+t)}\right)$ -Quasihyponormal for any  $0 < s < 1, 0 < t < 1$ .

**Proof:**

If  $T$  is a  $p$ -Quasihyponormal operator, then we have

$$T^* \left( (T^*T)^p - (TT^*)^p \right) T \geq 0$$

This implies that

$$|T|^p U^* \left( |T|^{2p^2} - (U|T|^{2p} U^*)^p \right) U |T|^p \geq 0$$

$$|T|^p U^* \left( |T|^{2p^2} - (U|T|^{2p^2} U^*) \right) U |T|^p \geq 0$$

This is equivalent to

$$|T|^p \left( U^* |T|^{2p^2} U - |T|^{2p^2} \right) |T|^p \geq 0$$

Thus on  $R(|T|^p)$ ,

$$U^* |T|^{2p^2} U \geq |T|^{2p^2}$$

$$U^* |T|^{2p^2} U \geq |T|^{2p^2} \geq U |T|^{2p^2} U^*$$

Let

$$A = U^* |T|^{2p^2} U, B = |T|^{2p^2}, C = U |T|^{2p^2} U^*$$

$$r = \frac{t}{2p^2}, u = \frac{2s}{2p^2}, q = \frac{1}{2(s+t)},$$

Since in Furuta's inequality

$$\left( 1 + \frac{t}{p^2} \right) 2(s+t) \geq \frac{s}{p^2} + \frac{t}{p^2}$$

and  $2(s+t) \geq 1$ ,

We have on  $R(\tilde{T}_{s,t})$ ,

$$\begin{aligned} \left( \tilde{T}_{s,t}^* \tilde{T}_{s,t} \right)^{\frac{1}{2(s+t)}} &= \left( |T|^t U^* |T|^{2s} U |T|^t \right)^{\frac{1}{2(s+t)}} \\ &= \left( B^{\frac{t}{2p^2}} A^{\frac{2s}{2p^2}} B^{\frac{t}{2p^2}} \right)^{\frac{1}{2(s+t)}} \\ &\geq B^{\left( \frac{2t}{2p^2} + \frac{2s}{2p^2} \right) \left( \frac{1}{2(s+t)} \right)} \\ &\geq B^{\left( \frac{s+t}{p^2} \right) \left( \frac{1}{2(s+t)} \right)} \\ &\geq B^{\frac{1}{2p^2}} \\ &\geq |T| \end{aligned} \quad \text{-----(1)}$$

Similarly,

$$\text{Since } \left( 1 + \frac{s}{p^2} \right) 2(s+t) \geq \frac{t}{p^2} + \frac{s}{p^2} \text{ and}$$

$2(s+t) \geq 1$ , we have

$$\begin{aligned} \left( \tilde{T}_{s,t} \tilde{T}_{s,t}^* \right)^{\left( \frac{1}{2(s+t)} \right)} &= \left( |T|^s U |T|^{2t} U^* |T|^s \right)^{\left( \frac{1}{2(s+t)} \right)} \\ &= \left( B^{\frac{s}{2p^2}} C^{\frac{2t}{2p^2}} B^{\frac{s}{2p^2}} \right)^{\left( \frac{1}{2(s+t)} \right)} \\ &\leq B^{\left( \frac{s+t}{p^2} \right) \left( \frac{1}{2(s+t)} \right)} \\ &\leq |T| \end{aligned} \quad \text{-----(2)}$$

By (1) and (2) we have

$$\left( \tilde{T}_{s,t}^* \tilde{T}_{s,t} \right)^{\left( \frac{1}{2(s+t)} \right)} \geq |T| \geq \left( \tilde{T}_{s,t} \tilde{T}_{s,t}^* \right)^{\left( \frac{1}{2(s+t)} \right)}$$

On  $R(\tilde{T}_{s,t})$

$$\tilde{T}_{s,t}^* \left( \left( \tilde{T}_{s,t}^* \tilde{T}_{s,t} \right)^{\left( \frac{1}{2(s+t)} \right)} - \left( \tilde{T}_{s,t} \tilde{T}_{s,t}^* \right)^{\left( \frac{1}{2(s+t)} \right)} \right) \tilde{T}_{s,t} \geq 0$$

Hence  $\tilde{T}_{s,t}$  is  $\left( \frac{1}{2(s+t)} \right)$ -Quasihyponormal

**Theorem: 2**

Let  $T = U|T|^p$  be the polar decomposition of a  $p$ -Quasihyponormal operator and  $U$  is unitary. For  $0 < r, t < 1, r \geq t$  and Let  $q = \max\{p+r+t, 2\}$  and  $\tilde{T}_{r,t} = |T|^r U |T|^{r-t}$ . Then  $\tilde{T}_{r,t}$  is  $q$ -Quasihyponormal.

**Proof:**

If  $T$  is a  $p$ -quasihyponormal operator, then we have

$$U^* |T|^{2p^2} U \geq |T|^{2p^2} \geq U |T|^{2p^2} U^*$$

Let

$$A = U^* |T|^{2p^2} U, B = |T|^{2p^2}, C = U |T|^{2p^2} U^*$$

$$q = \max \{p+r+t, 2\}$$

$$\begin{aligned} (\tilde{T}_{r,t} * \tilde{T}_{r,t})^q &= (|T|^{r-t} U^* |T|^{2t} U |T|^{r-t})^{\left(\frac{1}{q}\right)} \\ &= \left( B^{\frac{r-t}{2p^2}} A^{\frac{2t}{2p^2}} B^{\frac{r-t}{2p^2}} \right)^{\left(\frac{1}{q}\right)} \\ &\geq \left( B^{\frac{r-t}{2p^2}} B^{\frac{2t}{2p^2}} B^{\frac{r-t}{2p^2}} \right)^{\left(\frac{1}{q}\right)} \\ &= B^{\left(2\frac{r-t}{2p^2} + \frac{2t}{2p^2}\right)\left(\frac{1}{q}\right)} \\ &\geq B^{\left(\frac{r-t}{p^2} + \frac{t}{p^2}\right)\left(\frac{1}{q}\right)} \\ &= B^{\frac{r}{p^2 q}} \\ &\geq |T|^{\frac{2r}{q}} \text{------(3)} \end{aligned}$$

Since

$$\left(1 + 2\frac{r-t}{2p^2}\right)q \geq \frac{2t}{2p^2} + 2\frac{r-t}{2p^2} \text{ and } q \geq 1$$

Similarly

$$\begin{aligned} (\tilde{T}_{r,t} \tilde{T}_{r,t}^*)^q &= (|T|^t U |T|^{2(r-t)} U^* |T|^t)^{\left(\frac{1}{q}\right)} \\ &= \left( B^{\frac{t}{2p^2}} C^{\frac{2(r-t)}{2p^2}} B^{\frac{t}{2p^2}} \right)^{\left(\frac{1}{q}\right)} \\ &\leq \left( B^{\frac{r-t}{2p^2}} B^{\frac{2t}{2p^2}} B^{\frac{r-t}{2p^2}} \right)^{\left(\frac{1}{q}\right)} \\ &= B^{\left(2\frac{r-t}{2p^2} + \frac{2t}{2p^2}\right)\left(\frac{1}{q}\right)} \\ &\leq B^{\left(\frac{r-t}{p^2} + \frac{t}{p^2}\right)\left(\frac{1}{q}\right)} \\ &= B^{\frac{r}{p^2 q}} \leq |T|^{\frac{2r}{q}} \text{------(4)} \end{aligned}$$

Since

$$\left(1 + 2\frac{t}{2p^2}\right)q \geq \frac{2(r-t)}{2p^2} + 2\frac{t}{2p^2} \text{ and } q \geq 1$$

From (3) and (4) we have

$$(\tilde{T}_{r,t} * \tilde{T}_{r,t})^q \geq |T|^{\frac{2r}{q}} \geq (\tilde{T}_{r,t} \tilde{T}_{r,t}^*)^q$$

Hence on  $R(\tilde{T}_{r,t})$

$$(\tilde{T}_{r,t} * \tilde{T}_{r,t})^q \geq (\tilde{T}_{r,t} \tilde{T}_{r,t}^*)^q$$

This implies that

$$\tilde{T}_{r,t}^* \left( (\tilde{T}_{r,t} * \tilde{T}_{r,t})^q - (\tilde{T}_{r,t} \tilde{T}_{r,t}^*)^q \right) \tilde{T}_{r,t} \geq 0$$

Hence  $\tilde{T}_{r,t}$  is  $q$ -quasihyponormal.

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