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Sufficient Condition for Complete Graphs and Hamiltonian Graphs

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Abstact:In 1856, Hamiltonian introduced the Hamiltonian Graph where a Graph which is covered all the vertices without repetition and end with starting vertex. In this paper I would like to prove that every Complete Graph 'G' having $n \geq 5$ vertices, such that n is odd. If for all pairs of nonadjacent vertices u, v one has $d_u + d_v \geq n - 2$, then G has a Hamiltonian path.

Key Words : Graph, Complete Graph, Bipartite Graph Hamiltonian Graph

1. Introduction:

The origin of graph theory started with the problem of Koinsber bridge, in 1735.

This problem lead to the concept of Eulerian Graph. Euler studied the problem of Koinsberg bridge and constructed a structure to solve the problem called Eulerian graph.

In 1840, A.F Mobius gave the idea of complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems.

The concept of tree, (a connected graph without cycles was implemented by Gustav Kirchhoff in 1845, and he employed graph theoretical ideas in the calculation of currents in electrical networks or circuits.

In 1852, Thomas Gutherie found the famous four color problem.

Then in 1856, Thomas. P. Kirkman and William R.Hamilton studied cycles on polyhydra and invented the concept called Hamiltonian graph by studying trips that visited certain sites exactly once.

In 1913, H.Dudeney mentioned a puzzle problem. Eventhough the four color problem was invented it was solved only after a century by Kenneth Appel and Wolfgang Haken.

This time is considered as the birth of Graph Theory.

Caley studied particular analytical forms from differential calculus to study the trees. This had many implications in theoretical chemistry. This lead to the invention of enumerative graph theory.

Any how the term "Graph" was introduced by Sylvester in 1878 where he drew an analogy between "Quantic invariants" and covariants of algebra and molecular diagrams.

In 1941, Ramsey worked on colorations which lead to the identification of another branch of graph theory called extremel graph theory.

In 1969, the four color problem was solved using computers by Heinrich. The study of asymptotic graph connectivity gave rise to random graph theory.

In 1971 R.Halin introduced an example of minimally 3-connected Graphs.

1.1 Definition: A graph – usually denoted G(V,E) or G = (V,E) – consists of set of vertices V together with a set of edges E. The number of vertices in a graph is usually denoted *n* while the number of edges is usually denoted *m*.

1.2 Definition: Vertices are also known as nodes, points and (in social networks) as actors, agents or players.

1.3 Definition: Edges are also known as lines and (in social networks) as ties or links. An edge

e = (u,v) is defined by the unordered pair of vertices that serve as its end points.

1.4 Example: The graph depicted in Figure 1 has vertex set V={a,b,c,d,e.f} and edge set

E = {(a,b),(b,c),(c,d),(c,e),(d,e),(e,f)}.

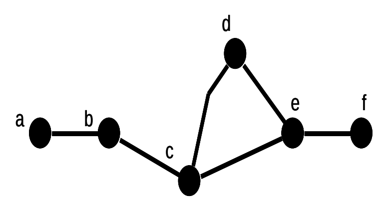


Figure .

1. 5 Definition: Two vertices *u* and *v* are *adjacent* if there exists an edge (*u*,*v*) that connects them.

1.6 Definition: An edge (u,v) is said to be *incident* upon nodes u and v.

1.7 Definition: An edge e = (u,u) that links a vertex to itself is known as a *self-loop* or *reflexive* tie.

1.8 Definition: Every graph has associated with it an *adjacency matrix*, which is a binary *n*×*n* matrix A in which $a_{ij} = 1$ and $a_{ji} = 1$ if vertex vi is adjacent to vertex vj, and aij = 0 and aji = 0 otherwise. The natural graphical representation of an adjacency matrix is a table, such as shown below.

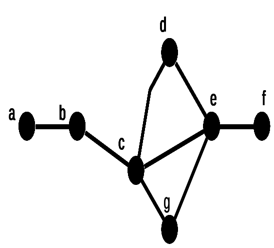
	a	b	c	d	e	f
a	0	1	0	0	0	0
b	1	0	1	0	0	0
c	0	1	0	1	1	0
d	0	0	1	0	1	0
e	0	0	1	1	0	1
f	0	0	0	0	1	0

Adjacency matrix for graph in Figure .

1.9 Definition: Examining either Figure 1 or given adjacency Matrix, we can see that not every vertex is adjacent to every other. A graph in which all vertices are adjacent to all others is said to be *complete*.

1.10 Definition: While not every vertex in the graph in Figure 1 is adjacent, one can construct a sequence of adjacent vertices from any vertex to any other. Graphs with this property are called *connected*.

1.11 Note: Reachability. Similarly, any pair of vertices in which one vertex can reach the other via a sequence of adjacent vertices is called *reachable*. If we determine reachability for every pair of vertices, we can construct a reachability matrix R such as depicted in Figure 2. The matrix R can be thought of as the result of applying transitive closure to the adjacency matrix A.



Figure

1.12 Definition : A walk is closed if $v_o = v_n$.*degree* of the vertex and is denoted d(v).

1.13 Definition : A *tree* is a connected graph that contains no cycles. In a tree, every pair of points is connected by a unique path. That is, there is only one way to get from A to B.

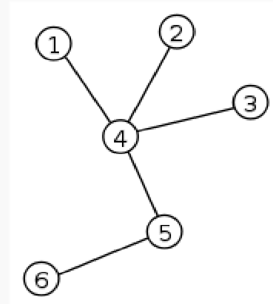


Figure: A labeled tree with 6 vertices and 5 edges

1.14 Definition: A *spanning tree* for a graph *G* is a sub-graph of *G* which is a tree that includes every vertex of *G*.

1.15 Definition: The length of a walk (and therefore a path or trail) is defined as the number of edges it contains. For example, in Figure 3, the path *a*,*b*,*c*,*d*,*e* has length 4.

1.16 Definition: The number of vertices adjacent to a given vertex is called the *degree* of the vertex and is denoted d(v).

1.17 Definition : In the mathematical field of graph theory, a bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

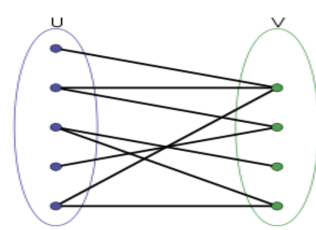


Figure : Example of a bipartite graph.

1.18 Definition : An Eulerian circuit in a graph G is circuit which includes every vertex and every edge of G. It may pass through a vertex more than once, but because it is a circuit it traverse each edge exactly once. A graph which has an Eulerian circuit is called an Eulerian graph. An Eulerian path in a graph G is a walk which passes through every vertex of G and which traverses each edge of G exactly once

1.19 Example : Königsberg bridge problem: The city of Königsberg (now Kaliningrad) had seven bridges on the Pregel River. People were wondering whether it would be possible to take a walk through the city passing exactly once on each bridge. Euler built the representative graph, observed that it had vertices of odd degree, and proved that this made such a walk impossible. Does there exist a walk crossing each of the seven bridges of Königsbergexactlyonce

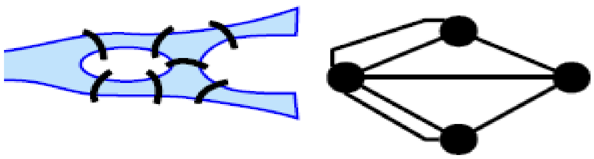


Figure : K¨onigsberg problem

2. **Hamiltonian Graphs, Complete Graphs Relation Between them :**

2.1 Definition : A Hamilton circuit is a path that visits every vertex in the graph exactly once and return to the starting vertex. Determining whether such paths or circuits exist is an NP-complete problem. In the diagram below, an example Hamilton Circuit would be

2.2 Example :

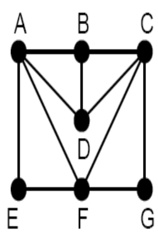


Figure:Hamilton Circuit would be AEFGCDBA.

2.3Theorem : Let $G = (V, E)$ be a Complete graph with $n \geq 5$ vertices, such that n is odd. If for all pairs of nonadjacent vertices u, v one has $d_u + d_v \geq n - 2$, then G has a Hamiltonian path.

Proof : Let $P = \langle x_1, x_2, \ldots, x_k \rangle$ be a longest path of G of length $k - 1$.
If P is a Hamiltonian path, we are done.
So,
Assume otherwise.
Clearly, $(x_1, x_k) \notin E$ because
Otherwise,
Let $G = (V, E)$ be a Complete graph with n vertices and P a longest path in G. If P is contained in a cycle then P is a Hamiltonian path , we are done.
Now, we must have[redacted]$d_{x1}[P^-] = d_{xk}[P^-] = 0$, since, otherwise, P is part of a longer path,
a contradiction.
Therefore, we must have $d_{x1} = d_{x1}[P]$ and $d_{xk} = d_{xk}[P]$.

Now, since x_1, x_k are nonadjacent, we must have

$d_{x1} + d_{xk} \geq n - 2$.

Now, if we assume that $d_{x1} + d_{xk} < k - 1$, then we must have $k - 2 \geq n - 2$;

i.e, $k \geq n$, implying that P is a Hamiltonian path.

Therefore, assume that $d_{x1} + d_{xk} \geq k - 1$.

Since we cannot allow a crossover edge,

If we want to maximize the total degree sum of $d_{x1} + d_{xk}$,

we can have configurations similar to one of three configurations, namely, Config-1, Config-2, and Config-3 shown in Figures 2 to 5.

It can be verified easily that any other configuration will result in $d_{x1} + d_{xk} < k - 1$ and according to the argument presented above P then would have to be a Hamiltonian path.

Note carefully that in both Config-1 and Config-2 we have $d_{x1} + d_{xk} \geq k - 1$.

He main properties of Config-1 and Config-2 are listed as the following facts.

And Config-3 is a combination of Config-1 and Config-2.

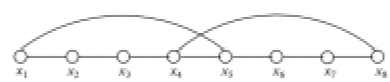
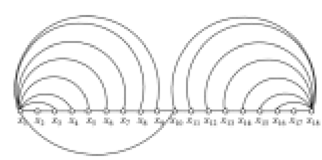
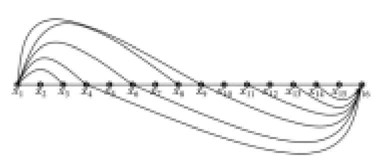


Figure 1 : Crossover edges.



<u>Figure 2</u> :Config-1.



<u>Figure 5</u> :Config-3: a combination of Config-1 and Config-2.

Fact 2. In Config-1, we have $N_{x1} = \{x_j | 1 < j \leq r\}$, $N_{xk} = \{x_j | r \leq j < k\}$, such that $N_{x1} \cap N_{xk} = \{x_r\}$.

Fact 3. In Config-2, if k is odd, then we have $N_{x1} = N_{xk} = \{x_j | j \text{ is even}\}$.

On the other hand, if k is even, then we have $N_{x1} = \{x_j | j \text{ is even and } j \neq k\}$ and

$N_{xk} = N_{x1} \cup \{x_{k-1}\}$.

Now, we **consider two cases** as follows.

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***Case 1* ($d_{x1} + d_{xk} > k - 1$) :**

From Figures 2 to 4 it is easy to see that if $d_{x1} + d_{xk} \geq k$, then we definitely will have a crossover edge resulting in a cycle containing another path P' such that

$|P| = |P'|$.

Therefore, Let $G = (V, E)$ be a Complete graph with n vertices and P a longest path in G. If P is contained in a cycle then P is a Hamiltonian path, it follows that the length of a longest path cannot be $k - 1$, a contradiction.

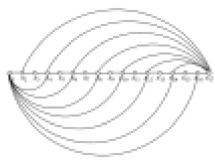

Figure 4 :Config-2: *k* is odd.

***Case 2* ($d_{x1} + d_{xk} = k - 1$) :**

Since $(x_1, x_k) \notin E$, we have $d_{x1} + d_{xk} \geq n - 2$.

Therefore, $k - 1 \geq n - 2 \Longrightarrow k \geq n - 1$.

Now, if $k > n - 1$, we are done.

So, assume that $k = n - 1$.

Then we have a vertex y such that $y \in V - V[P]$. Now, for y, we cannot have

$(y, x_1), (y, x_k) \in E$ because, then, we have a longer path than P,

a contradiction.

Now we must have $d_y + d_{x1} \geq n - 2$ and $d_y + d_{xk} \geq n - 2$.

Then, we have the following

$$\begin{aligned} d_{x_1} + d_{x_k} + 2 \times d_y &\geq 2 \times (n-2) \\ \Longrightarrow 2 \times d_y + k - 1 &\geq 2n - 4 \\ \Longrightarrow 2 \times d_y + n - 1 - 1 &\geq 2n - 4 \\ \Longrightarrow 2 \times d_y + n &\geq 2n - 2 \\ \Longrightarrow 2 \times d_y &\geq n - 2 \\ \Longrightarrow d_y &\geq \frac{(n-2)}{2} \\ \Longrightarrow d_y &\geq \frac{n}{2} - 1. \end{aligned}$$

Now, since $n \geq 5$ and d_y cannot be a fractional value, we must have $d_y \geq 2$.

Now we have two cases.

Case 2.a (Config-1).

In this case we have a configuration similar to Figure 2.

Now, let $x_i, x_j \in N_y$. Assume without loss of generality that $j > i$. If $j = i + 1$, then we easily get a Hamiltonian path $P' = \langle x_1, x_2, \ldots, x_i, y, x_j, x_{j+1}, \ldots, x_k \rangle$ and we are done.

So, assume that $j = i + l$, $l > 1$.

Now, recall that, in this case, there exists a vertex x_r, $r \notin \{1, k\}$ such that

$(x_r, x_1), (x_r, x_k) \in E$ (Fact 2).

Now, we have three subcases.

***Case 2.a.1* ($r \leq i < j < k$).**

From Fact 2, it is clear that $(x_k, x_{i+1}) \in E$. Therefore, we get a Hamiltonian path.

and we are done.

***Case 2.a.2* ($1 < i < j \leq r$).**

This is symmetrical to Case 2.a.1.

***Case 2.a.3* ($1 < i < r < j$).**

Again, from Fact 2, it is clear that $(x_k, x_{j-1}) \in E$. Therefore, we get a Hamiltonian path

P" = $\langle x_1, x_2 \ldots\ldots\ldots, x_i, y, x_j, x_{j+1}, \ldots\ldots\ldots, x_k, x_{i+1}, x_{i+2}, \ldots\ldots, x_{j-1} \rangle$

and we are done.

***Case 2.a.2* ($1 < i < j \leq r$).**

This is symmetrical to Case 2.a.1.

***Case 2.a.3* ($1 < i < r < j$).**

Again, from Fact 2, it is clear that $(x_k, x_{j-1}) \in E$.

Therefore, we get a Hamiltonian path

P"' = $\langle x_1, x_2 \ldots\ldots\ldots, x_i, y, x_j, x_{j+1}, \ldots\ldots\ldots, x_k, x_{j-1}, x_{j-2}, \ldots, x_{i+1} \rangle$

and we are done.

Case 2.b (Config-2).

Since n is odd, in this case, we have k is even. Hence, we have a configuration similar to

Figure 3. Recall that we have $d_y \geq (n/2) - 1$. Since n is odd and d_y cannot be a fractional number, we must have $d_y \geq ((n + 1)/2) - 1$. In other words, we have $d_y \geq ((k + 2)/2) - 1 = k/2$.

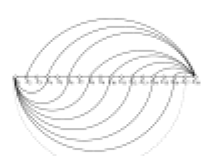

Figure 3 : Config-2: *k* is even (the dotted edge is nonexistent).

First of all, if we have either of the edges (y, x_1), $(y, x_k) \in E$, we are done.

So assume otherwise. Furthermore, if we have $(y, x_{k-1}) \in E$, then we get a Hamiltonian path $P' = \langle x_1, x_2, \ldots, x_{k-2}, x_k, x_{k-1}, y\rangle$ and we are done. So, assume otherwise.

So, $x_1, x_{k-1}, x_k \notin N_y$. Therefore, we have $k - 3$ vertices as candidates for membership in N_y.

Now, since $n \geq 5$, we have $k = n - 1 \geq 4$. Therefore, we must have $d_y \geq 2$.

Now, let $x_i, x_j \in N_y$. Without the loss of generality assume that $j > i$.

Clearly, if $j = i + 1$, we are done,

since we get a Hamiltonian path

$P' = \langle x_1, x_2, \ldots, x_i, y, x_j, x_{j+1}, \ldots, x_k\rangle$.

So, assume that $j = i + l$, $l > 1$. Since $k - 3$ is an odd number, it follows that $d_y \leq (k - 2)/2 = (k/2) - 1$. This contradicts our deduction above that $d_y \geq k/2$.

***Case 2.c (Config-3)*.**

As mentioned above Config-3 is a combination of Config-1 and Config-2. Recall that in this case our assumption is $d_{x1} + d_{xk} = k - 1$.

Hence we have $d_y \geq (n/2) - 1$. Clearly, $x_1, x_k \notin N_y$ because then we get a longer path, a contradiction.

It is easy to verify that this would force two consecutive vertices $x_i, x_{i+1} \in N_y$ for $2 \leq i \leq n - 2$.

Then, again we get a longer path simply including the subpath $\langle x_i, y, x_{i+1}\rangle$, which leads us to a contradiction.

And this completes our proof.

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