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Data Estimation at Unmeasured Positions Using Measured Receptance Frequency Response Function

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Abstract:The dynamic responses of finite element model do not match with those of the experimental results due to modeling and measurement errors. It is impractical to collect the data for the full set of DOFs of a dynamic system. And there are some cases to have difficulty in measuring the response such as slope. This work presents an analytical method to estimate the frequency response function (FRF) at the unmeasured nodes of the actual system. The method is derived by minimizing the discrepancy between the analytical and actual FRF data sets dividing into the real and imaginary parts. The validity of the proposed method is investigated in a numerical application to estimate the unmeasured slope FRF data of a beam.

Keyword: Pseudo inverse, Frequency response function, Data expansion, Cost function, Measurement

1. Introduction

An accurate dynamic finite element model of a structure is very important for structural design and analysis. The frequency response of the finite element model does not correspond to experimental measurements due to the existence of uncorrected noise content and the modeling errors. If a discrepancy between the two is found, the analytical results should be modified based on the experimental measurements. It is improper to collect a full set of dynamic responses, and all response data cannot be obtained from vibration testing. The problem can be overcome by data reduction or data expansion, i.e., either reduce the system matrices to the number of measured DOFs or conversely expand the measured response vectors to the full size of the finite element model matrices.

Most of the utilized expansion techniques involve the use of the finite element model as a mechanism to complete the unmeasured DOFs from the experimental model. Imregun and Ewins [1] presented mode shape expansion methods to estimate the modal displacements at unmeasured coordinates using the incomplete set of measured data together with a corresponding finite element model. Levine-West et al. [2], [3] evaluated the robustness and reliability of the Guyan static expansion method, the Kidder dynamic method, the Procrustes method, and the penalty method along with least-squares minimization techniques. Dubbaka and Houghton [4] provided a correction method to modify the measured mode shapes by restricting the orthonormality and by applying the constraint rules. Based on a dynamic modal expansion that minimizes the residual error in the eigenvalue equation for each measured mode, Kenneth and Francois [5] presented an algorithm for expanding the measured mode shapes obtained from modal testing of the full set of DOFs of a corresponding finite element model.

Measured FRF data provide certain advantages over modal data. The FRF data contain the effect of the vibration modes and dynamic characteristics. A full set of FRF data should be expanded using measured data.

An FRF matrix of the full set of DOFs measured by experiments is used to predict parameter matrices of the dynamic system. It has been reported [6] that FRF data provide more information than modal data, as the latter are extracted from a very limited frequency range related to resonance. Friswell and Penny [7] proposed an approach to reduce the model order so that the stated estimation process reduces to a least squares problem based on the FRFs. Utilizing FRF data measured at specific positions, with dofs less than that of the system, as constraints to describe a damaged system, Rahmatalla et al. [8] identified parameter matrices such as mass, stiffness and damping matrices of the system, and provided a damage identification method from their variations.

Considering the inconsistency between the analytical and experimental responses due to the modeling and measurement errors, the FRF data should be corrected for the subsequent design and analysis. This work presents an analytical method to estimate the FRF at the unmeasured nodes of the actual system. The method is derived by minimizing the discrepancy between the analytical and actual FRF data sets dividing into the real and imaginary parts. The validity of the proposed method is investigated in the FRF data expansion to estimate the unmeasured slope FRF data of a beam.

2. Formulation

Dynamic responses are expressed in the time domain and frequency domain. For linear systems there is little loss of information going from the time domain to the frequency domain. The FRFs in the frequency domain are measured instead of displacement and force individually. The dynamic responses can be initially obtained from a mathematical model such as a finite element method. The dynamic behaviour is approximately discretized for *n* DOFs and can be described by the equations of motion as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ denote the $n \times n$ mass and stiffness matrices, $\mathbf{u} = [u_1 \quad u_2 \quad \cdots \quad u_n]^T$, $\mathbf{C} \in R^{n \times n}$ is the damping matrix, and $\mathbf{f}(t)$ is the $n \times 1$ excitation vector.

Inserting $\mathbf{u} = \mathbf{U}e^{i\Omega t}$ and $\mathbf{f} = \mathbf{F}e^{i\Omega t}$ into Eq. (1) and expressing it in the frequency domain, it follows that

$$\mathbf{D}_{ini}(\Omega)\mathbf{U}(\Omega) = \mathbf{F}(\Omega) \tag{2}$$

where the dynamic stiffness matrix of the initial dynamic model, $\mathbf{D}_{ini}$, is written by

$$\mathbf{D}_{ini}(\Omega) = \mathbf{K} - \Omega^2\mathbf{M} + i\Omega\mathbf{C} \tag{3}$$

where the subscript *ini* represents the initial finite element model, $i = \sqrt{-1}$, Ω denotes the excitation frequency, $\mathbf{F}(\Omega) = [F_1 \quad F_2 \quad \cdots \quad F_n]^T$ and $\mathbf{U}(\Omega) = [U_1 \quad U_2 \quad \cdots \quad U_n]^T$ represent the Fourier transform of the force and response vectors $\mathbf{f}$ and $\mathbf{u}$, respectively. Equation (2) is valid for an excitation frequency Ω . The $n \times n$ FRF matrix is defined as $\mathbf{H}_{ini}$,

$$\mathbf{H}_{ini} \equiv \left[\mathbf{K} - \Omega^2\mathbf{M} + i\Omega\mathbf{C}\right]^{-1} \tag{4}$$

where $H_{ini,ij}$ in the FRF matrix denotes the displacement response measured at location *i* due to the unit impulse force input at location *j*.

The FRF matrix of the initial model is expressed by dividing into the real and imaginary parts as

$$\mathbf{H}_{ini}(\Omega) = \mathbf{H}_{ini,re}(\Omega) + i\mathbf{H}_{ini,im}(\Omega) \tag{5}$$

The FRF matrix corresponding to a full set of DOFs of the dynamic system can be numerically obtained by dividing into the real and imaginary parts.

If variable parameters of mass, stiffness and damping are randomly added due to the modeling and measurement errors, the dynamic equation of Eq. (2) is modified as

$$\mathbf{D}_{\mathrm{mod}}(\Omega)\mathbf{U}(\Omega) = \mathbf{F}(\Omega) \tag{6}$$

where $\mathbf{D}_{\mathrm{mod}}(\Omega)$ denotes the dynamic stiffness matrix of the modified system by

$$\mathbf{D}_{\mathrm{mod}}(\Omega) = (\mathbf{K} + \Delta\mathbf{K}) + i\Omega(\mathbf{C} + \Delta\mathbf{C}) - \Omega^2(\mathbf{M} + \Delta\mathbf{M}) \tag{7}$$

And the FRF matrix of the modified system, $\mathbf{H}_{\mathrm{mod}}$, is written by

$$\mathbf{U}(\Omega) = \mathbf{H}_{\mathrm{mod}}\mathbf{F}(\Omega) \tag{8}$$

where

$$\mathbf{H}_{\mathrm{mod}}(\Omega) = \left[(\mathbf{K} + \Delta\mathbf{K}) + i\Omega(\mathbf{C} + \Delta\mathbf{C}) - \Omega^2(\mathbf{M} + \Delta\mathbf{M})\right]^{-1} .$$

The modified FRF matrix is expressed by the sum of the real part and imaginary part as

$$\mathbf{H}_{\mathrm{mod}}(\Omega) = \mathbf{H}_{\mathrm{mod},re}(\Omega) + i\mathbf{H}_{\mathrm{mod},im}(\Omega) \tag{9}$$

The FRF data at unmeasured positions should be estimated by expanding the measurement data. Assume the FRF data of the displacement responses due to the unit impulse at $m(m < n)$ different locations less than the number of total DOFs of the system are collected. The $m \times m$ measured FRF data set, $\mathbf{H}_{\mathrm{mod}}$, can be expressed by

$$\mathbf{AU} = \mathbf{BF} \tag{10}$$

where $\mathbf{A}$ is an $m \times n$ Boolean matrix to define the displacement measurement locations and $\mathbf{B}$ represents the $m \times n$ FRF matrix including *m* receptances due to the *m* impulse excitations, $\mathbf{U}$ denotes an $n \times m$ displacement response matrix, $\mathbf{B} = \mathbf{U}^T$, and $\mathbf{F}$ is the $n \times m$ excitation matrix to define the unit impulse at *m* locations. The matrices $\mathbf{B}$ and $\mathbf{U}$ are also expressed in the real and imaginary terms of FRF matrix as

$$\mathbf{B} = \mathbf{B}_{re} + i\mathbf{B}_{im}, \quad \mathbf{U} = \mathbf{U}_{re} + i\mathbf{U}_{im} \tag{11}$$

This work presents an analytical method to expand $\mathbf{U}$ or $\mathbf{B}$ including the experimentally measured FRF data to a full set of DOFs. The measured FRF data represent as the constraint condition for the expansion. The expansion is performed by dividing into the real and imaginary parts, and a least square method utilizing cost functions of

$$J = \left\|\mathbf{H}_{ini,re}^{-1}\left(\mathbf{U}_{re} - \mathbf{H}_{ini,re}\right)\right\| + \left\|\mathbf{H}_{ini,im}^{-1}\left(\mathbf{U}_{im} - \mathbf{H}_{ini,im}\right)\right\| \tag{12}$$

is applied. In order to utilize Eq. (10) into Eq. (12), Eq. (10) is modified as

$$\mathbf{A}\mathbf{H}_{ini,re}^{1/2}\mathbf{H}_{ini,re}^{-1/2}\mathbf{U}_{re} = \mathbf{B}_{re}\mathbf{F} \tag{13}$$

$$\mathbf{A}\mathbf{H}_{ini,im}^{1/2}\mathbf{H}_{ini,im}^{-1/2}\mathbf{U}_{im} = \mathbf{B}_{im}\mathbf{F} \tag{14}$$

Letting $\mathbf{R}_1 = \mathbf{A}\mathbf{H}_{ini,re}^{1/2}$ and $\mathbf{R}_2 = \mathbf{A}\mathbf{H}_{ini,im}^{1/2}$, and solving the result with respect to $\mathbf{H}_{ini,re}^{-1/2}\mathbf{U}_{re}$ and $\mathbf{H}_{ini,im}^{-1/2}\mathbf{U}_{im}$, respectively, we obtain that

$$\mathbf{H}_{ini,re}^{-1/2}\mathbf{U}_{re} = \mathbf{R}_1^{+}\mathbf{B}_{re}\mathbf{F} + \left(\mathbf{I} - \mathbf{R}_1^{+}\mathbf{R}_1\right)\mathbf{y}_1 \tag{15}$$

$$\mathbf{H}_{ini,im}^{-1/2}\mathbf{U}_{im} = \mathbf{R}_2^{+}\mathbf{B}_{im}\mathbf{F} + \left(\mathbf{I} - \mathbf{R}_2^{+}\mathbf{R}_2\right)\mathbf{y}_2 \tag{16}$$

where the superscript '+' indicates the Moore-Penrose inverse, and $\mathbf{y}_1$ and $\mathbf{y}_2$ are arbitrary matrices. Equations (15) and (16) have an infinite number of solutions due to the presence of the arbitrary matrices $\mathbf{y}_1$ and $\mathbf{y}_2$. The expansion is carried out by providing the condition to minimize the cost function of Eq. (12) of all solutions of Eqs. (15) and (16). Utilizing the condition to minimize Eq. (12) into Eqs. (15) and (16), and solving it with respect to the arbitrary matrices, respectively, they are derived as

$$\mathbf{y}_1 = \left(\mathbf{I} - \mathbf{R}_1^{+}\mathbf{R}_1\right)\left(\mathbf{I} - \mathbf{R}_1^{+}\mathbf{B}_{re}\mathbf{F}\right) + \mathbf{R}_1^{+}\mathbf{R}_1\mathbf{s}_1 \tag{17}$$

$$\mathbf{y}_2 = \left(\mathbf{I} - \mathbf{R}_2^{+}\mathbf{R}_2\right)\left(\mathbf{I} - \mathbf{R}_2^{+}\mathbf{B}_{im}\mathbf{F}\right) + \mathbf{R}_2^{+}\mathbf{R}_2\mathbf{s}_2 \tag{18}$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ are also arbitrary matrices. Utilizing Eqs. (17) and (18) into Eqs. (15) and (16), respectively, and arranging them, the following equations are obtained as

$$\mathbf{H}_{ini,re}^{-1/2}\mathbf{U}_{re} = \mathbf{R}_1^{+}\mathbf{B}_{re}\mathbf{F} + \left(\mathbf{I} - \mathbf{R}_1^{+}\mathbf{R}_1\right) \tag{19}$$

$$\mathbf{H}_{ini,im}^{-1/2}\mathbf{U}_{im} = \mathbf{R}_2^{+}\mathbf{B}_{im}\mathbf{F} + \left(\mathbf{I} - \mathbf{R}_2^{+}\mathbf{R}_2\right) \tag{20}$$

Arranging Eqs. (19) and (20), and pre-multiplying both sides of the resulting equations by $\mathbf{H}_{ini,re}^{1/2}$ and $\mathbf{H}_{ini,im}^{1/2}$, respectively, the $n \times n$ resulting FRF matrices to be expanded to a full set of DOFs can be written by

$$\mathbf{U}_{re} = \mathbf{H}_{ini,re}^{1/2} + \mathbf{H}_{ini,re}^{1/2}\left(\mathbf{A}\mathbf{H}_{ini,re}^{1/2}\right)^{+}\left(\mathbf{B} - \mathbf{A}\mathbf{H}_{ini,re}^{1/2}\right) \tag{21}$$

$$\mathbf{U}_{im} = \mathbf{H}_{ini,im}^{1/2} + \mathbf{H}_{ini,im}^{1/2}\left(\mathbf{A}\mathbf{H}_{ini,im}^{1/2}\right)^{+}\left(\mathbf{B} - \mathbf{A}\mathbf{H}_{ini,im}^{1/2}\right) \tag{22}$$

The numerical results of Eqs. (21) and (22) are expressed by the real and imaginary parts as

$$\mathbf{U}_{re} = \mathbf{U}_{re,re} + i\mathbf{U}_{re,im} \tag{23}$$

$$\mathbf{U}_{im} = \mathbf{U}_{im,re} + i\mathbf{U}_{im,im} \tag{24}$$

The resulting FRF matrix can be expressed by summing Eqs. (23) and (24) as

$$\mathbf{H}_{\mathrm{mod}} = \left(\mathbf{U}_{re,re} + \mathbf{U}_{im,re}\right) + i\left(\mathbf{U}_{re,im} + \mathbf{U}_{im,im}\right) \tag{25}$$

Equation (25) represents the $n \times n$ FRF matrix expanded from the $m \times m$ measured FRF data set. It is shown that the proposed method can be explicitly utilized in expanding the FRF data.

3. A numerical experiment

A fixed-end beam, shown in Fig. 1, was considered to investigate the validity of the proposed method to expand the FRF data. The nodal points and the members for the finite element beam model are numbered in the figure. Each node has two DOFs of the vertical displacement and slope, and the beam is modeled by 49 nodes and 98 DOFs besides the boundary conditions. A 5-m long beam was modeled using 50 beam elements of $100\mathrm{mm}$. The elastic modulus of the beam was $2.0 \times 10^5\ \mathrm{MPa}$, and its unit mass and gross cross-section were $7{,}860 \times 10^{-6}\ \mathrm{kg/m}$ and $200\mathrm{mm} \times 100\mathrm{mm}$, respectively. The stiffness and mass matrices can be established using the mechanical properties of the beam. The damping matrix was assumed as the Rayleigh damping to be expressed by the stiffness matrix and the proportional constant 0.0001.

The analytical FRF matrix by the given parameter matrices doesn't coincide with the one of the actual beam due to the modeling and measurement errors. For this numerical experiment, the simulated parameter matrices of the actual beam were established as follows:

$$\mathbf{M}_{\mathrm{mod}} = \mathbf{M}_{ini}\left(\mathbf{I} + \alpha_m \sigma_m\right)$$

$$\mathbf{K}_{\mathrm{mod}} = \mathbf{K}_{ini}\left(\mathbf{I} + \alpha_k \sigma_k\right)$$

$$\mathbf{C}_{\mathrm{mod}} = \mathbf{C}_{ini}\left(\mathbf{I} + \alpha_c \sigma_c\right)$$

where $\mathbf{I}$ denotes the identity matrix, α_m, α_k and α_c represent the relative magnitudes of the error for the mass, stiffness and damping matrices, respectively, and σ_m, σ_k and σ_c are $n \times n$ random number matrices in the range $[-1,\ \ 1]$. Figure 2 exhibits their variation deviated from the analytical parameter matrices. The error magnitudes included in the parameter matrices were established as $\alpha_m = \alpha_k = \alpha_c = 0.01$.

It's not easy to measure the slope responses of the beam due to the external excitation. Using the proposed method, this study was performed to estimate the receptance FRF data corresponding to all slope DOFs from the measured displacement FRF data. Figure 3 represents the receptance FRF magnitude due to the impulse excitation at node 5. In the figure, the dashed and solid lines indicate the estimated and actual value, respectively. As shown in the plots, the displacement and estimated slope FRF receptance curves exhibit the similar values at the resonance frequency region as the actual values. However, the values at the antiresonance frequency represent large difference between two data sets. It should be overcome by increasing the number of measured DOFs.

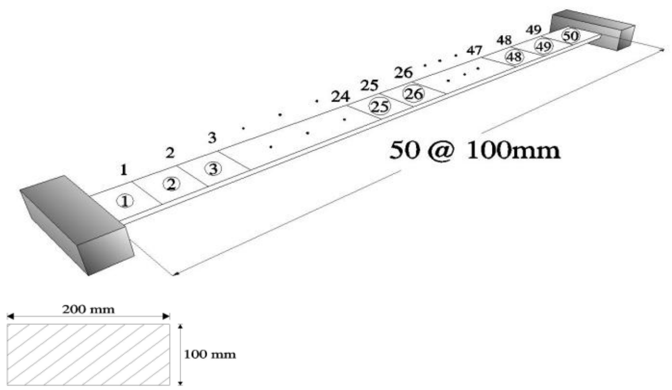
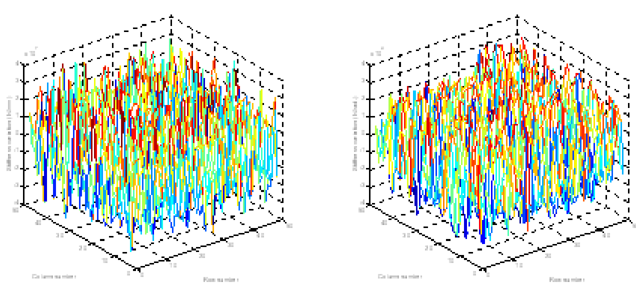
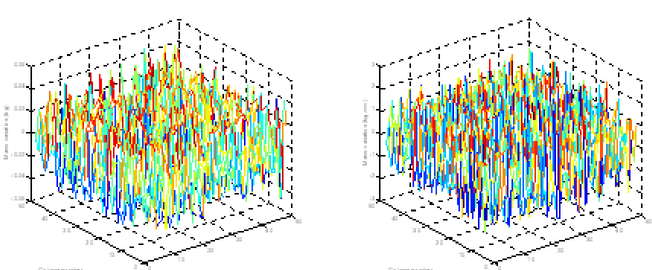


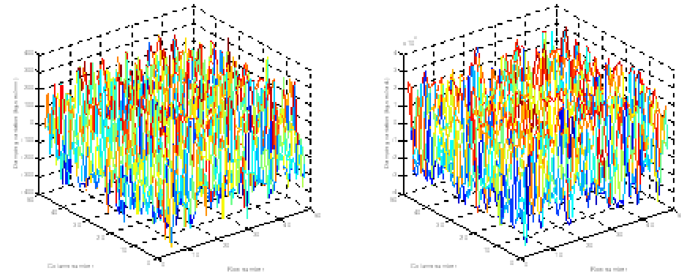
Fig. 1 Finite element model of a fixed-end beam



(a) (b)

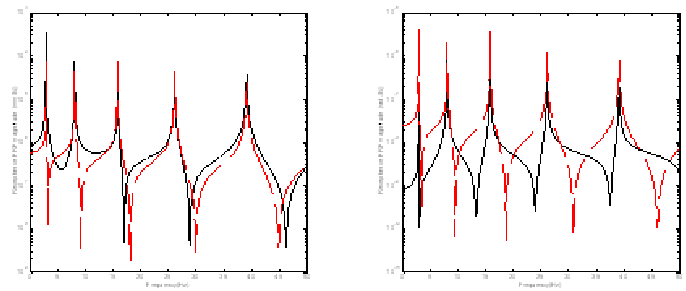


(c) (d)

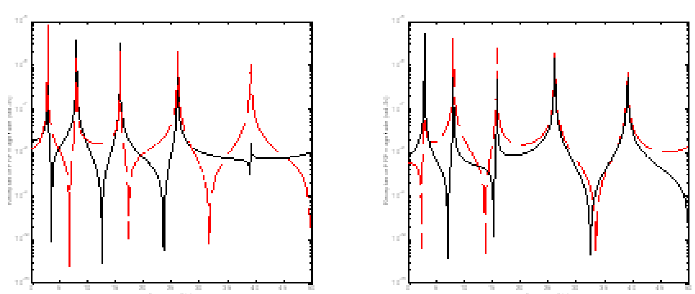


(e) (f)

Fig. 2 Variation in parameter matrices: (a) stiffness corresponding to the displacements, (b) stiffness corresponding to the slopes, (c) mass corresponding to the displacements, (d) mass corresponding to the slopes, (e) damping corresponding to the displacements, (f) damping corresponding to the slopes



(a) (b)



(c) (d)

Fig. 3 Receptance FRF data due to the impulse force at node 5: (a) displacement FRF at node 1, (b) slope FRF at node 3, (c) slope FRF at node 12, (d) slope FRF at node 11

4. Conclusion

Analytically calculated FRF data do not coincide with the actual test data due to the construction and measurement errors. And it is not practical to collect the FRF data at all DOFs. This work presented a data expansion method to estimate the FRF data at unmeasured nodes. The displacement and estimated slope FRF receptance curves exhibit the similar values at the resonance frequency region as the actual values. However, the values at the antiresonance frequency represent large difference between two data sets. It should be overcome by increasing the number of measured DOFs.

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REFERENCES

i. *Imregun, M. and Ewins, D.J., An investigation into mode shape expansion techniques, IMAC 11th International Modal Analysis Conference, Kissimee, FL, 1993.*

ii. *Levine-West, M., Kissil, A. and Milman, M., Evaluation of mode shape expansion techniques on the micro-precision interferometer truss, IMAC 12th International Modal Analysis Conference, Honolulu, HI, 1994.*

iii. *Levine-West, M.B., Milman, M. and Kissil A. 1996. Mode shape expansion techniques for prediction: Experimental evaluation, AIAA Journal, 34, 821-829.*

iv. *Dubbaka, K.R. and Houghton, J.R., Correction of experimental mode shapes, IMAC 14th International Modal Analysis Conference, Dearborn, MI, 1996.*

v. *Kenneth, F.A. and Francois, H., Dynamic mode shape expansion using mass orthogonality constraints, IMAC 18th International Modal Analysis Conference, Sanantonio, TX, 2000.*

vi. *Lee, U. and Shin, J. 2002. A frequency response function-based structural damage identification method, Comput. Struct., 80(2), 117-132.*

vii. *Friswell, M.I. and Penny, J.E.T. 1990. Updating model parameters from frequency domain data via reduced order models, Mech. Syst. Signal Pr., 4(5), 377-391.*

viii. *Rahmatalla, S., Eun, H.C. and Lee, E.T. 2012. Damage detection from the variation of parameter matrices estimated by incomplete FRF data. Smart Structures and Systems, 9, 55-70.*