International Journal of Scientific Engineering and Technology
Volume No.6 Issue No.1, pp: 45-49

ISSN:2277-1581
01 Jan. 2017

Estimation of Parameters in the Growth Curve Model with a Linearly Structured Covariance Matrix – A Simulation Study

Cassien Habyarimana[1,]*, Martin Singull[2], Joseph Nzabanita[3]
[1]Integrated Polytechnic Regional Centre, Rwanda; [2]Linköping University, Sweden;
[3]University of Rwanda, Rwanda
*Corresponding author. Email address: habyarimanacassien@gmail.com

Abstract: In this paper, the implementation of algorithm proposed in (Nzabanita, J., et al. 2012) for some known linear structures on the covariance matrix Σ is performed and simulations for different sample sizes are repeated many times. For these simulations, the percentages of non positive definite estimates are produced, and the linear structures are identified and classified.

Keywords: Growth curve model, Estimator, Linearly structured covariance matrix, Positive definite matrix.

1. Introduction

A growth curve is an empirical model of the evolution of a quantity over time. Growth curves are widely used in biology for quantities such as population size, body height or biomass (Pan, et al., 2002). In mathematical statistics, growth curves are often modelled as being continuous stochastic processes, e.g. as being sample paths that almost surely solve stochastic differential equations (Seber, G. A. F. and Wild, C. J., 1989). The growth curve model has an important application in many areas such as medicine, pharmacy, natural sciences, social sciences, etc. The growth curve model has been extensively studied over many years and it was introduced in (Pothoff, R. and Roy, S., 1964). With improvement the growth curve model was extended and studied by many authors, for instance (Nzabanita, J., et al., 2012; Seid Hamid, J. and von Rosen, D., 2006; Verbyla, A. and Venables, W., 1988; Yokoyama, T., 1996; Srivastava, M. S. and von Rosen, D., 1999).

The estimation of parameters in the growth curve model, when the covariance matrix has some specific linear structure has been discussed by some authors, for example (Nzabanita, J., et al., 2012; Ohlson, M. and von Rosen, D., 2010). In Ohlson, M. and von Rosen, D. (2010), when the classical growth curve model with linearly structured covariance matrix is considered, a suggested estimation procedure gives explicit and consistent estimators of both the mean and the covariance matrix, and in (Nzabanita, J., et al.,2012), when the extended growth curve model with two terms and a linearly structured covariance matrix is studied, also a suggested estimation procedure gives explicit and consistent estimators of the covariance matrix.

The idea is first to estimate the covariance matrix when finding the inner product in a regression space and thereafter re-estimate it when it should be interpreted as a covariance matrix. This idea was first considered by (Ohlson, M. and von Rosen, D., 2010) and is exploited by decomposing the residual space, the orthogonal complement to the design space, into orthogonal subspaces. Studying residuals obtained from projections of observations on these subspaces yields explicit consistent estimator of the covariance matrix. However, through simulation for some linear structures on the extended growth curve model, it was noted in (Nzabanita, J., et al. 2012) that the estimates of the covariance matrix may not be positive definite for small sample sizes whereas it is always positive definite for some other structures for moderate sample sizes. Hence, in this paper we studied how the problem of non-positive definiteness for the estimates of the covariance matrix $\mathbf{\Sigma}$ for some linearly structured covariance matrices would be identified.

The model we study may be defined in the following way:

Let $X : p \times n$ be an observation matrix, $A_i : p \times q_i$ be a within individual design matrix, $B_i : q_i \times k_i$ be a parameter matrix, $C_i : k_i \times n$ be a between individual design matrix, for $i=1, 2$, $r(C_1) + p \leq n$ and $\mathcal{C}(\mathbf{C}'_2) \subseteq \mathcal{C}(\mathbf{C}'_1)$ where $r(\cdot)$ and $\mathcal{C}(\cdot)$ represent the rank and the column space of a matrix, respectively. The extended growth curve model with two terms and a linearly structured covariance matrix is defined as follows,

$$X = A_1 B_1 C_1 + A_2 B_2 C_2 + E \qquad (1)$$

where the columns of $\boldsymbol{E}$ are assumed to be independently distributed as a p-variate normal distribution with mean zero and a positive definite covariance matrix $\mathbf{\Sigma} = (\sigma_{ij})_{i,j=1}^{p}$ i.e., $\boldsymbol{E} \sim MN_{p,n}(\boldsymbol{0}, \boldsymbol{\Sigma}, \boldsymbol{I}_n)$.

The covariance matrix $\mathbf{\Sigma}$ has some linear structure. The matrices $\boldsymbol{A}_i$ and $\boldsymbol{C}_i$ are known matrices whereas matrices $\boldsymbol{B}_i$ and $\boldsymbol{\Sigma}$ are unknown parameter matrices.

2. Estimators in the growth curve model with a linearly structured covariance matrix

In this section, we derived estimators for the covariance matrix in the extended Growth Curve model with a linearly structured covariance matrix.

2.1. Linearly structured matrix and linear structures

Definition 2.1 (Linearly structured matrix) *A matrix* $\mathbf{\Sigma} = \sigma_{ij}$ *is linearly structured if the only linear structure between the elements is given by* $\sigma_{ij} = \sigma_{kl}$ *and there exists at* *least one (i,j)≠(k,l) so that* $|\sigma_{ij}| = |\sigma_{kl}|$.

The linear structures for the covariance matrices emerged naturally in statistical applications and they are in the statistical literature for some years ago. More details on the examples of structures are developed in subsection 2.2.

2.2. Different linear structures for the covariance matrix

Covariance matrix with zeros

An example of a covariance matrix with zeros is given by

$$\Sigma_{WZ} = \begin{pmatrix} \sigma_1^2 & 0 & \sigma_{13} & 0 \\ 0 & \sigma_2^2 & 0 & \sigma_{24} \\ \sigma_{13} & 0 & \sigma_3^2 & \sigma_{34} \\ 0 & \sigma_{24} & \sigma_{34} & \sigma_4^2 \end{pmatrix}.$$

Banded covariance structure

The banded covariance structure can for example be

$$\Sigma_B = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & 0 & 0 \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & 0 \\ 0 & \sigma_{23} & \sigma_3^2 & \sigma_{34} \\ 0 & 0 & \sigma_{34} & \sigma_4^2 \end{pmatrix}.$$

Toeplitz and circular Toeplitz covariance structure

Toeplitz covariance structure is given by

$$\Sigma_T = \begin{pmatrix} \sigma^2 & \rho_1 & \rho_2 & \rho_3 \\ \rho_1 & \sigma^2 & \rho_1 & \rho_2 \\ \rho_2 & \rho_1 & \sigma^2 & \rho_1 \\ \rho_3 & \rho_2 & \rho_1 & \sigma^2 \end{pmatrix} \quad \text{or} \quad \Sigma_T = \begin{pmatrix} \sigma_1^2 & \rho_1 & \rho_2 & \rho_3 \\ \rho_1 & \sigma_2^2 & \rho_1 & \rho_2 \\ \rho_2 & \rho_1 & \sigma_3^2 & \rho_1 \\ \rho_3 & \rho_2 & \rho_1 & \sigma_4^2 \end{pmatrix}$$

and the circular Toeplitz covariance structure is given by

$$\Sigma_{CT} = \begin{pmatrix} \sigma^2 & \rho_1 & \rho_2 & \rho_1 \\ \rho_1 & \sigma^2 & \rho_1 & \rho_2 \\ \rho_2 & \rho_1 & \sigma^2 & \rho_1 \\ \rho_1 & \rho_2 & \rho_1 & \sigma^2 \end{pmatrix} \quad \text{or} \quad \Sigma_{CT} = \begin{pmatrix} \sigma^2 & \rho_1 & \rho_2 & \rho_2 & \rho_1 \\ \rho_1 & \sigma^2 & \rho_1 & \rho_2 & \rho_2 \\ \rho_2 & \rho_1 & \sigma^2 & \rho_1 & \rho_2 \\ \rho_2 & \rho_2 & \rho_1 & \sigma^2 & \rho_1 \\ \rho_1 & \rho_2 & \rho_2 & \rho_1 & \sigma^2 \end{pmatrix}.$$

Intraclass covariance structure

The intraclass covariance structure looks like

$$\Sigma_{IC} = \begin{pmatrix} \sigma^2 & \rho_0 & \rho_0 & \rho_0 \\ \rho_0 & \sigma^2 & \rho_0 & \rho_0 \\ \rho_0 & \rho_0 & \sigma^2 & \rho_0 \\ \rho_0 & \rho_0 & \rho_0 & \sigma^2 \end{pmatrix}.$$

Compound symmetry (type I and II)

Votaw, D. F. (1948) extended the intraclass model to a model with blocks called compound symmetry, type I and II and are follows

$$\Sigma_{CS-I} = \begin{pmatrix} \alpha & \beta & \beta & \beta \\ \beta & \gamma & \delta & \delta \\ \beta & \delta & \gamma & \delta \\ \beta & \delta & \delta & \gamma \end{pmatrix} \quad \text{and} \quad \Sigma_{CS-II} = \begin{pmatrix} \alpha & \beta & \kappa & \sigma \\ \beta & \alpha & \sigma & \kappa \\ \kappa & \sigma & \gamma & \delta \\ \sigma & \kappa & \delta & \gamma \end{pmatrix}.$$

2.3. Estimators in an extended growth curve model

Considering the extended growth curve model given by equation (1), $X=A_1B_1C_1+A_2B_2C_2+E$, but with $E \sim MN_{p,n}(0,\Sigma^{(s)},I_n)$ where $\Sigma^{(s)}$ is a linearly structured covariance matrix. Assuming that matrices A_i, C_i for $i=1, 2$ are of full rank and that $\mathcal{C}(\mathbf{A}_1)\cap\mathcal{C}(\mathbf{A}_2)=\{\mathbf{0}\}$. The main estimator of the linearly structured covariance matrix $\mathbf{\Sigma}^{(s)}$ proposed by (Nzabanita, J., et al. 2012) equals

$$\text{vec}\widehat{\mathbf{\Sigma}}^{(s)} = \mathbf{T}^{+}\left((\mathbf{T}^{+})'\widehat{\mathbf{\Phi}}'\widehat{\mathbf{\Phi}}\mathbf{T}^{+}\right)^{-1}(\mathbf{T}^{+})'\widehat{\mathbf{\Phi}}'\text{vec}\mathbf{S}. \qquad (2)$$

where $\mathbf{S}$ is the sum of squares matrix given by $\mathbf{S} = \widehat{\mathbf{H}}_1\widehat{\mathbf{H}}_1' + \widehat{\mathbf{H}}_2\widehat{\mathbf{H}}_2' + \widehat{\mathbf{H}}_3\widehat{\mathbf{H}}_3'$ and $\widehat{\mathbf{\Phi}} = (\mathrm{n}-\mathrm{r}_1)\mathbf{I} + (\mathrm{r}_1-\mathrm{r}_2)\widehat{\mathbf{T}}_1\otimes\widehat{\mathbf{T}}_1 + \mathrm{r}_2\widehat{\mathbf{T}}_2\oplus\widehat{\mathbf{T}}_2$ with $\widehat{\mathbf{T}}_2 = \mathbf{I} - \mathbf{P}_{\mathbf{A}_1,\widehat{\mathbf{\Sigma}}_1^{(s)}} - \mathbf{P}_{\widehat{\mathbf{T}}_1\mathbf{A}_2,\widehat{\mathbf{\Sigma}}_2^{(s)}}$ and $\widehat{\mathbf{T}}_1 = \mathbf{I} - \mathbf{P}_{\mathbf{A}_1,\widehat{\mathbf{\Sigma}}_1^{(s)}}$ where $\otimes$ denotes the Kronecker product and **vecM** denotes the vectorization of **M**. $\mathbf{T}^{+}$ is the Moore-Penrose inverse of **T** and **T** is a matrix such that $vec\mathbf{\Sigma}^{(s)}(\mathbf{K})=\mathbf{T}vec\mathbf{\Sigma}^{(s)}$ where $vec\mathbf{\Sigma}^{(s)}(\mathbf{K})$ is a columnwise vectorized form of $\mathbf{\Sigma}^{(s)}$. $\hat{H}_1$, $\hat{H}_2$ and $\hat{H}_3$ are projectors given by $\hat{H}_1 = X(I - P_{C_1'})$, $\hat{H}_2 = (I - P_{A_1\hat{\Sigma}_1^{(s)}})X(P_{C_1'} - P_{C_2'})$ and $\hat{H}_3 = (I - P_{A_1\hat{\Sigma}_1^{(s)}} - P_{\hat{T}_1A_2\hat{\Sigma}_2^{(s)}})XP_{C_2'}$.

The above projectors are obtained from the whole space decomposition according to the within and between individual designs illustrating the mean and residual spaces, see Figure 1.

$\hat{\Sigma}_1^{(s)}$ and $\hat{\Sigma}_2^{(s)}$ are the estimators of $\Sigma^{(s)}$ obtained by considering only the residual $\hat{H}_1$ and by considering both $\hat{H}_1$ and $\hat{H}_2$ respectively (see Figure 1) and are given by

$$\text{vec}\widehat{\mathbf{\Sigma}}_1^{(s)} = \frac{1}{\mathrm{n}-\mathrm{r}_1}\mathbf{T}^{+}((\mathbf{T}^{+})'\mathbf{T}^{+})^{-1}(\mathbf{T}^{+})'\text{vec}\mathbf{S}_1. \qquad (3)$$

and

$$\text{vec}\widehat{\mathbf{\Sigma}}_2^{(s)} = \mathbf{T}^{+}\left((\mathbf{T}^{+})'\widehat{\mathbf{\Upsilon}}'\widehat{\mathbf{\Upsilon}}\mathbf{T}^{+}\right)^{-1}(\mathbf{T}^{+})'\widehat{\mathbf{\Upsilon}}'\text{vec}\left(\widehat{\mathbf{H}}_1\widehat{\mathbf{H}}_1' + \widehat{\mathbf{H}}_2\widehat{\mathbf{H}}_2'\right). \quad (4)$$

where $\widehat{\mathbf{\Upsilon}} = (\mathrm{n}-\mathrm{r}_1)\mathbf{I} + (\mathrm{r}_1-\mathrm{r}_2)\widehat{\mathbf{T}}_1\otimes\widehat{\mathbf{T}}_1$.

The estimators developed above have some properties like unbiasedness and consistency, these properties should be proved else refers to the paper where in (Nzabanita, J., et al. 2012), it has been shown that

- The estimator $\hat{\Sigma}_1^{(s)}$ given in (3) is a consistent estimator of $\Sigma^{(s)}$, i.e., $\hat{\Sigma}_1^{(s)} \xrightarrow{p} \Sigma^{(s)}$.
- The estimator $\hat{\Sigma}_2^{(s)}$ given in (4) is a consistent estimator of $\Sigma^{(s)}$, i.e., $\hat{\Sigma}_2^{(s)} \xrightarrow{p} \Sigma^{(s)}$.
- The estimator $\hat{\Sigma}^{(s)}$ given in (2) is a consistent estimator of $\Sigma^{(s)}$, i.e., $\hat{\Sigma}^{(s)} \xrightarrow{p} \Sigma^{(s)}$.

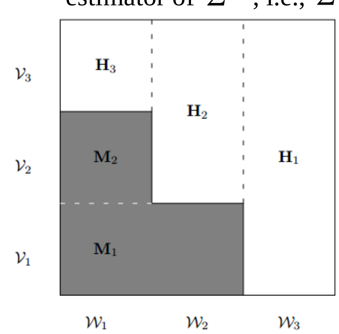

Figure 1: Decomposition of the whole space according to the

within and between individual designs illustrating the mean and residual spaces where $\mathcal{V}_1 = \mathcal{C}_{\Sigma^{(s)}}(A_1), \mathcal{V}_2 = \mathcal{C}_{\Sigma^{(s)}}(T_1 A_2)$, $\mathcal{V}_3 = (\mathcal{C}_{\Sigma^{(s)}}(A_1) + \mathcal{C}_{\Sigma^{(s)}}(T_1 A_2))^{\perp}$, $\mathcal{W}_1 = \mathcal{C}(\mathbf{C}'_2)$, $\mathcal{W}_2 = (\mathcal{C}(\mathbf{C}'_1) \cap \mathcal{C}(\mathbf{C}'_2))^{\perp}$, $\mathcal{W}_3 = \mathcal{C}(\mathbf{C}'_1)$, $\mathbf{M}_1 = \mathbf{P}_{A_1\Sigma^{(s)}} X P_{C'_1}$, $\mathbf{M}_2 = \mathbf{P}_{T_1 A_2 \Sigma^{(s)}} X P_{C'_2}$ where $^{\perp}$ denotes the orthogonal complement.

3. Simulation Studies

3.1. Description of Scenarios

Simulations were done with Matlab code implementing the formula (5) for every linear structure discussed in section 2. In each simulation a sample of observations was randomly generated from a p-variate growth curve model. Sample sizes from *n=10*, *20,...,*(small), *n=100*(moderate) to *n=500*(large sample size) were considered. The simulations are repeated *N=1000* times and the following design matrices were used

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}, A_2 = \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \end{pmatrix} \quad \text{for} \quad p = 4 \quad \text{and}$$

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix}, A_2 = \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{pmatrix} \quad \text{for} \quad p = 5 \quad \text{and}$$

$$C_1 = \begin{pmatrix} ones(1, n/2) & zeros(1, n/2) \\ zeros(1, n/2) & ones(1, n/2) \end{pmatrix},$$

$C_2 = (zeros(1, n/2), \ ones(1, n/2))$ where n is the sample size, A_i and C_i are the within and between individual design matrices respectively for $i=1, 2$.

3.2. Results and discussions

For every structure, the averaged estimates of the covariance matrix for small sample size $n = 10$ is reported, the graphs of the percentages of non-positive definite estimates are plotted, a graph and a table that summarize all information are reported.

For covariance matrix with zeros

When the covariance matrix with zeros is considered, the averaged estimate of

$$\boldsymbol{\Sigma} = \begin{pmatrix} 16 & 0 & 3 & 0 \\ 0 & 4 & 0 & 5 \\ 3 & 0 & 9 & 2 \\ 0 & 5 & 2 & 10 \end{pmatrix}$$

for $n = 10$ is given by

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 257.0860 & 0 & -88.0042 & 0 \\ 0 & 44.6585 & 0 & -33.2569 \\ -88.0042 & 0 & 64.6777 & 2.7536 \\ 0 & -33.2569 & 2.7536 & 64.7439 \end{pmatrix}$$

For small sample size $n = 10$ the averaged estimate is not closed to the proposed value of $\boldsymbol{\Sigma}$.

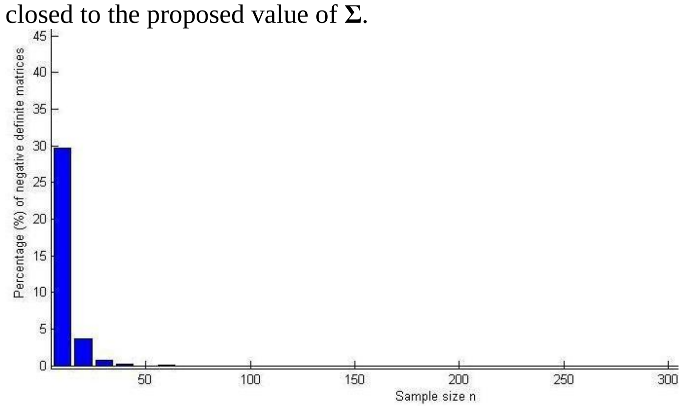

Figure 2: Percentage of non-positive estimates of $\boldsymbol{\Sigma}$ for covariance matrix with zeros

Banded covariance structure (with $p = 4$)

The averaged estimate of the positive definite covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 7 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 3 & 6 \end{pmatrix}$$

for $n = 10$ is given by

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 77.8090 & -15.6525 & 0 & 0 \\ -15.6526 & 54.3274 & 86.7390 & 0 \\ 0 & 86.7390 & 105.2802 & -16.9658 \\ 0 & 0 & -16.9658 & 114.3014 \end{pmatrix}$$

For small sample size $n = 10$ the averaged estimate is not closed to the proposed value of $\boldsymbol{\Sigma}$.

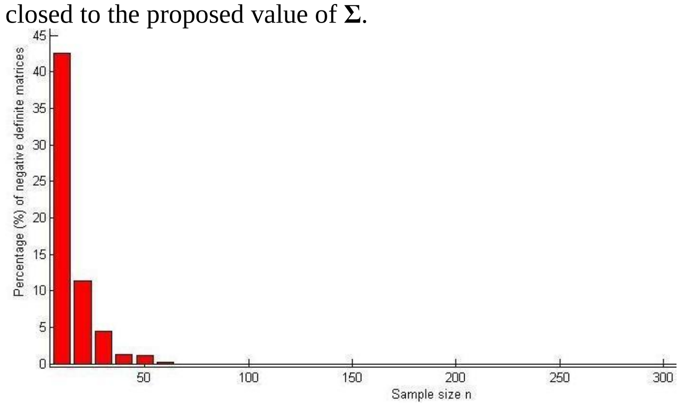

Figure 3: Percentage of non-positive estimates of $\boldsymbol{\Sigma}$ for banded covariance structure with p=4

Toeplitz covariance structure (with the same variances)

For this linear structure the estimate of the positive definite covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 4 & 1 & 2 & 3 \\ 1 & 4 & 1 & 2 \\ 2 & 1 & 4 & 1 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

for n = 10 is given by

$$\hat{\mathbf{\Sigma}} = \begin{pmatrix} 4.4566 & 1.3752 & 2.1237 & 3.3081 \\ 1.3752 & 4.4566 & 1.3752 & 2.1237 \\ 2.1237 & 1.3752 & 4.4566 & 1.3752 \\ 3.3081 & 2.1237 & 1.3752 & 4.4566 \end{pmatrix}$$

The obtained averaged estimate of $\mathbf{\Sigma}$ is closed to the true value.

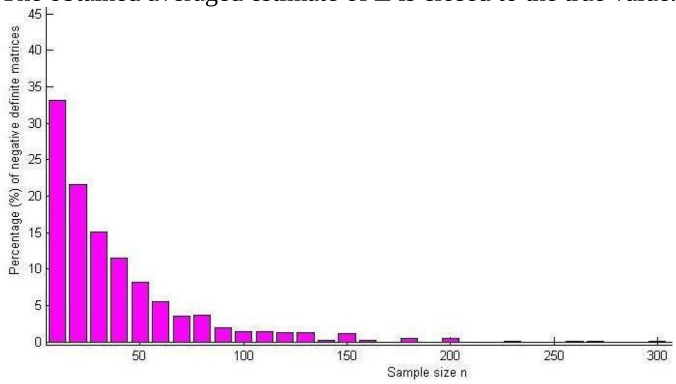

Figure 4: Percentage of non-positive estimates of $\mathbf{\Sigma}$ for toeplitz covariance structure with same variances

Toeplitz covariance structure (with different variances)

For this structure, the estimate of the positive definite covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} 4 & 1 & 2 & 3 \\ 1 & 5 & 1 & 2 \\ 2 & 1 & 6 & 1 \\ 3 & 2 & 1 & 7 \end{pmatrix}$$

for n = 10 is given by

$$\hat{\mathbf{\Sigma}} = \begin{pmatrix} 123.9509 & 0.1276 & -28.6401 & 83.8243 \\ 0.1276 & 30.0239 & 0.1276 & -28.6401 \\ -28.6401 & 0.1276 & 22.8943 & 0.1276 \\ 83.8243 & -28.6401 & 0.1276 & 149.9859 \end{pmatrix}$$

For small sample size n = 10 the averaged estimate of $\mathbf{\Sigma}$ is not closed to the proposed value of $\mathbf{\Sigma}$.

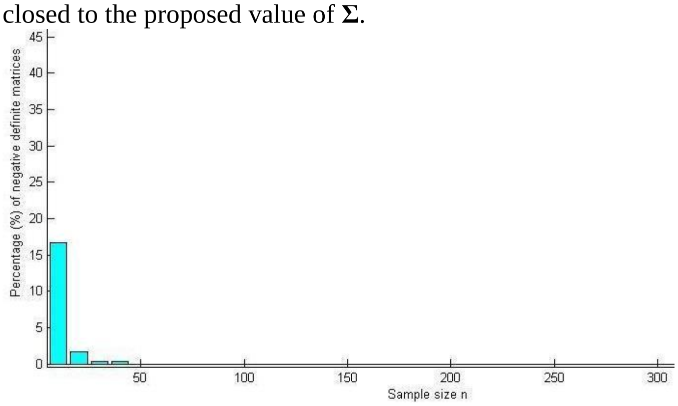

Figure 5: Percentage of non-positive estimates of $\mathbf{\Sigma}$ for toeplitz covariance structure with different variances

Circular Toeplitz covariance structure (with p = 4)

When this structure is considered, the estimate for $n=10$ of the positive definite covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} 4 & 1 & 2 & 1 \\ 1 & 4 & 1 & 2 \\ 2 & 1 & 4 & 1 \\ 1 & 2 & 1 & 4 \end{pmatrix} \text{ is } \hat{\mathbf{\Sigma}} = \begin{pmatrix} 3.9652 & 0.9617 & 1.9537 & 0.9617 \\ 0.9617 & 3.9652 & 0.9617 & 1.9537 \\ 1.9537 & 0.9617 & 3.9652 & 0.9617 \\ 0.9617 & 1.9537 & 0.9617 & 3.9652 \end{pmatrix}$$

The averaged estimate of $\mathbf{\Sigma}$ is closed to the true value.

Circular Toeplitz covariance structure (with p = 5)

When this structure is considered, the estimate of

$$\mathbf{\Sigma} = \begin{pmatrix} 4 & 1 & 2 & 2 & 1 \\ 1 & 4 & 1 & 2 & 2 \\ 2 & 1 & 4 & 1 & 2 \\ 2 & 2 & 1 & 4 & 1 \\ 1 & 2 & 2 & 1 & 4 \end{pmatrix} \text{ is } \hat{\mathbf{\Sigma}} = \begin{pmatrix} 3.9928 & 1.0134 & 2.0089 & 2.0089 & 1.0134 \\ 1.0134 & 3.9928 & 1.0134 & 2.0089 & 2.0089 \\ 2.0089 & 1.0134 & 3.9928 & 1.0134 & 2.0089 \\ 2.0089 & 2.0089 & 1.0134 & 3.9928 & 1.0134 \\ 1.0134 & 2.0089 & 2.0089 & 1.0134 & 3.9928 \end{pmatrix}$$

Here the averaged estimate of $\mathbf{\Sigma}$ is closed to the true value and this structure shows a zero percentage of non-positive definite estimates of $\mathbf{\Sigma}$ for all sample size n (see Table 1).

Intraclass covariance structure

When this structure is considered, the estimate for $n=10$ of

$$\mathbf{\Sigma} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \text{ is } \hat{\mathbf{\Sigma}} = \begin{pmatrix} 1.9930 & 0.9939 & 0.9939 & 0.9939 \\ 0.9939 & 1.9930 & 0.9939 & 0.9939 \\ 0.9939 & 0.9939 & 1.9930 & 0.9939 \\ 0.9939 & 0.9939 & 0.9939 & 1.9930 \end{pmatrix}$$

The averaged estimate of $\mathbf{\Sigma}$ is closed to the true value and this linear structure shows zero percentage of non-positive definite estimates of $\mathbf{\Sigma}$ for all sample size n (see Table 1).

Compound symmetric type I structure

When this structure is considered, the estimate for $n=10$ of

$$\mathbf{\Sigma} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 10 & 9 & 9 \\ 1 & 9 & 10 & 9 \\ 1 & 9 & 9 & 10 \end{pmatrix} \text{ is } \hat{\mathbf{\Sigma}} = \begin{pmatrix} 2.3202 & 0.7483 & 0.7483 & 0.7483 \\ 0.7483 & 13.5256 & 12.4115 & 12.4115 \\ 0.7483 & 12.4115 & 13.5256 & 12.4115 \\ 0.7483 & 12.4115 & 12.4115 & 13.5256 \end{pmatrix}$$

In this case, the averaged estimate of $\mathbf{\Sigma}$ is closed to the true value but this structure shows only a small percentage of non-positive definite estimates of the covariance matrix $\mathbf{\Sigma}$ for sample size around n = 10 (see Table 1).

Compound symmetric type II structure

For this structure the estimate of

$$\mathbf{\Sigma} = \begin{pmatrix} 5 & 2 & 3 & 1 \\ 2 & 5 & 1 & 3 \\ 3 & 1 & 6 & 4 \\ 1 & 3 & 4 & 6 \end{pmatrix}$$

for n = 10is given by

$$\hat{\mathbf{\Sigma}} = \begin{pmatrix} 37.0450 & -3.9938 & -26.6162 & 19.2260 \\ -3.9938 & 37.0450 & 19.2260 & -26.6162 \\ -26.6162 & 19.2260 & 63.9544 & -1.1993 \\ 19.2260 & -26.6162 & -1.1993 & 63.9544 \end{pmatrix}$$

For this linear structure the averaged estimate is not closed to the proposed value of $\mathbf{\Sigma}$ for small sample size n = 10. This structure shows a small percentage of non-positive definite matrices of the estimate of $\mathbf{\Sigma}$ for small sample size around n = 10 (see Table 1).

In summary, as said by (Nzabanita, J., et al. 2012), our results concluded that for some linear structures, the estimates of $\mathbf{\Sigma}$ may be positive definite or not for small sample size. The class of circular Toeplitz covariance, and intraclass covariance structures show 100% of positive definite estimates of $\mathbf{\Sigma}$ for all sample sizes, and compound symmetry (type I&II), the covariance matrix with zeros, the banded covariance and Toeplitz covariance (with the same/different variances) structures show a non-zero percentage of non-positive definite estimates of $\mathbf{\Sigma}$ for small and/or moderate sample sizes.

The Table 1 below shows the percentage of non-positive definite estimates of the linearly structured covariance matrix **Σ** for different sample sizes (n) and different linear structures (**LS**) for the extended growth curve model (**EGCM**) where **LS1** stands for the covariance matrix with zeros, **LS2** for the banded covariance structure, **LS3** for the Toeplitz covariance structure with the same variances, **LS4** for the Toeplitz covariance structure with the different variances, **LS5** for the circular Toeplitz covariance structure with p = 4, **LS6** for the circular Toeplitz covariance structure with p = 5, **LS7** for intraclass covariance structure (or uniform covariance structure), and **LS8** and **LS9** for the compound symmetry structure type I and II respectively. The figure 6 shows the classification of the linear structures.

Sample size (n)	Percentage of non positive definite estimates of Σ (%)								
	LS1	LS2	LS3	LS4	LS5	LS6	LS7	LS8	LS9
10	29.7	42.6	33.2	16.7	0	0	0	2.9	5.0
20	3.7	11.3	21.6	1.7	0	0	0	0	0
30	0.7	4.4	15.1	0.3	0	0	0	0	0
40	0.2	1.2	11.5	0.3	0	0	0	0	0
50	0	1.1	8.2	0	0	0	0	0	0
60	0.1	0.2	5.5	0	0	0	0	0	0
70	0	0	3.5	0	0	0	0	0	0
80	0	0	3.6	0	0	0	0	0	0
90	0	0	1.9	0	0	0	0	0	0
100	0	0	1.4	0	0	0	0	0	0
110	0	0	1.4	0	0	0	0	0	0
120	0	0	1.2	0	0	0	0	0	0
130	0	0	1.3	0	0	0	0	0	0
140	0	0	0.2	0	0	0	0	0	0
150	0	0	1.1	0	0	0	0	0	0
160	0	0	0.2	0	0	0	0	0	0
170	0	0	0	0	0	0	0	0	0
180	0	0	0.4	0	0	0	0	0	0
190	0	0	0	0	0	0	0	0	0
200	0	0	0.4	0	0	0	0	0	0
210	0	0	0	0	0	0	0	0	0
220	0	0	0	0	0	0	0	0	0
230	0	0	0.1	0	0	0	0	0	0
240	0	0	0	0	0	0	0	0	0
250	0	0	0	0	0	0	0	0	0
260	0	0	0.1	0	0	0	0	0	0
270	0	0	0.1	0	0	0	0	0	0
280	0	0	0	0	0	0	0	0	0
290	0	0	0	0	0	0	0	0	0
300	0	0	0.1	0	0	0	0	0	0

Table 1: Percentage of non-positive definite estimates of **Σ** for different sample sizes n and different linear structures for EGCM.

The linear structures are classified according to the graph of Figure 6 below

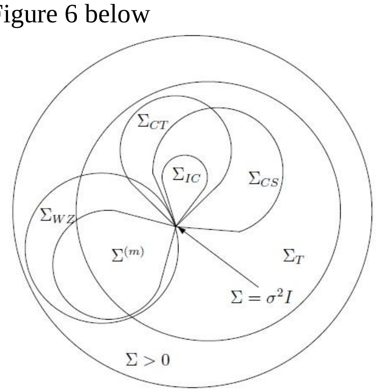

Figure 6: Different covariance structures. ($\mathbf{\Sigma_{WZ}}$= *with zeros*, $\mathbf{\Sigma^{(m)}}$= *banded*, $\mathbf{\Sigma_T}$= *Toeplitz*, $\mathbf{\Sigma_{CT}}$=*circular Toeplitz*, $\mathbf{\Sigma_{CS}}$= *compound symmetry* and $\mathbf{\Sigma_{IC}}$= *intraclass*)

4. Concluding remarks

In this paper, we implemented the algorithm proposed by (Nzabanita, J., et al. 2012). We identified and classified the structures that produce positive definite estimates for the linearly structured covariance matrix **Σ** where the class of circular Toeplitz covariance and intraclass covariance structures show 100% of positive definite estimates of **Σ** for all sample sizes.

Acknowledgement

I thank God the Almighty who has provided me the breath of life to enable the successful completion of this work and for His unconditional love. I would like to thank my supervisor Professor Martin Singull and my co-supervisor Dr. Joseph Nzabanita, immensely for helping throughout this work. Most importantly, I want to thank my family and my friends for being with me through thick and thin of my life and being a great motivation.

References

i. *Nzabanita, J., et al. (2012).Estimation of parameters in the extended growth curve model with a linearly structured covariance matrix. Acta et Commentationes Universitatis Tartuensis de Mathematica, 16(1): 13 − 32.*

ii. *Ohlson, M. and von Rosen, D. (2010). Explicit estimators of parameters in the growth curve model with linearly structured covariance matrices. Journal of Multivariate Analysis, 101: 1284 − 1295.*

iii. *Pan, et al. (2002). Growth curve models and statistical diagnostics. Springer Series in Statistics. New York: Springer-Verlag. ISBN 0 − 387 − 95053 − 2.*

iv. *Pothoff, R. and Roy, S. (1964). A generalized multivariate analysis of variance model useful especially for growth curve problems. Biometrika, 51: 313 − 326.*

v. *Seber, G. A. F. and Wild, C. J. (1989). Growth models. Nonlinear regression. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. New York: John Wiley & Sons, Inc. pp. 325 − 367.*

vi. *Seid Hamid, J. and von Rosen, D. (2006). Residuals in the extended growth curve model. Scandinavian Journal of Statistics, 33(1): 121 − 138.*

vii. *Srivastava, M. S. and von Rosen, D. (1999). Growth curve model. In: Multivariate Analysis, Design of Experiments, and Survey Sampling. (S. Ghosh, ed.), Statistics: Textbooks and Monographs 159, Dekker, New York, pp. 547 − 578.*

viii. *Verbyla, A. and Venables, W. (1988). An extension of the growth curve model. Biometrica.*

ix. *Votaw, D. F. (1948). Testing compound symmetry in a normal multivariate distribution. The Annals of Mathematical Statistics, 19(4): 447 − 473.*

x. *Yokoyama, T. (1996). Extended Growth Curve models with random-effects covariance structures. Communications in Statistics - Theory and Methods, 25(3): 571 − 584.*